Abstract

The dynamic stiffness matrix of a functionally graded beam (FGB) is developed using a higher order shear deformation theory. The material properties of the FGB are varied in the thickness direction based on a power-law. The kinetic and potential energies of the beam are formulated by accounting for a parabolic shear stress distribution. Hamilton's principle is used to derive the governing differential equations of motion in free vibration. The analytical expressions for axial force, shear force, bending moment and higher order moment at any cross-section of the beam are obtained as a by-product of the Hamiltonian formulation. The differential equations are solved in closed analytical form for harmonic oscillation. The dynamic stiffness matrix of the FGB is then constructed by relating the amplitudes of forces and displacements at the ends of the beam. The Wittrick-Williams algorithm is applied to the dynamic stiffness matrix of the FGB to compute its natural frequencies and mode shapes in the usual way after solving the eigenvalue problem. Finally, some conclusions are drawn.

Keywords: Free vibration, functionally graded beams, dynamic stiffness method, Wittrick-Williams algorithm, parabolic shear deformation beam theory.

1 Introduction

Functionally graded materials (FGM) are characterised by continuous transition of material properties as a function of position along certain directions. The gradual variation of material properties can be designed for specific function and applications. The analysis of structures made of FGM has attracted many researchers in recent years since the concept of FGM was first introduced in Japan around the mid-1980s. Beam as a load carrying member in a structure is widely used in aeronautical, civil, mechanical and other installations. It is a potential candidate for FGM application. Thus, the dynamic behaviour of the functionally graded beams

(FGB) has become an area of concentrated research. Some recent publications [1-8] are included in this paper. Applying the dynamic stiffness method (DSM), the free vibration analysis of FGB has earlier been investigated using the Bernoulli-Euler [7] and Timoshenko [8] beam theories. It is well known that the Bernoulli-Euler beam theory does not consider the shear stresses and associated strains in the cross-section and it assumes that the cross-section remains plane and perpendicular to the beam axis after deformation. By contrast, the Timoshenko beam theory is based on the first order shear deformation which takes into account the effects of shear deformation and rotatory inertia. Although the Timoshenko beam theory is a refinement over the Bernoulli-Euler theory, it is nevertheless, deficient because it assumes constant shear stress and shear strain distribution in the cross-section which clearly violates the free shear stress condition on the outer surface of the beam. A shape factor is often introduced in the Timoshenko beam theory, but its usage can sometimes be unsatisfactory and often controversial. The dynamic vibration behaviour of beam elements has been researched using a higher order beam theory [9, 10] which eliminates the ambiguity of using a fictitious shape factor. Thus the development of the dynamic stiffness method using a higher order shear deformation theory for free vibration analysis of FGB is important. This has not been attempted before and hence rightfully becomes the subject matter of this paper to investigate the free vibration behaviour of FGB in an accurate manner. The current investigation is no-doubt a significant step forward following the authors' recent contributions to the state-of-the-art using the DSM theories of Bernoulli-Euler [7] and Timoshenko [8] beams, respectively.

The DSM uses exact member theory based on frequency dependent shape functions obtained from the solution of governing differential equations of motion of structural elements in free vibration. Therefore, results for all natural frequencies and mode shapes are exact without making any approximation en route. Furthermore, results are also independent of the number of elements used in the analysis. This is of course, impossible in the conventional finite element method (FEM) and many other approximate methods. The DSM has always been distinctive and is probably the ultimate benchmark in free vibration analysis. Thus DSM can be used to validate the FEM and other approximate methods.

The development of the dynamic stiffness element of a functionally graded beam using a higher order shear deformation theory is of course, the main focus of this paper. Material properties are assumed to vary continuously in the beam thickness direction based on a power law distribution. First the kinetic and potential energies of the functionally graded beam are formulated using a higher order shear deformation theory which accounts for parabolic distribution of the transverse shear strains through the thickness of the beam. Then the governing differential equations of motion in free vibration and the natural boundary conditions are derived using Hamilton's principle which was facilitated by symbolic computation [11]. Next the differential equations are solved in closed analytical form for harmonic oscillation. By relating the amplitudes of forces to those of the displacements at the beam ends, the dynamic stiffness matrix is developed. Finally the Wittrick–Williams algorithm

[12] is applied to the ensuing dynamic stiffness matrix as the solution technique to yield the natural frequencies and mode shapes.

2 Theory

2.1 Derivation of the governing differential equations

A uniform rectangular cross section FGB is shown in Figure 1 in a right-handed Cartesian coordinate system. The beam has a length L, width b, and thickness h, with the mechanical properties of the beam: Young's modulus E, Poisson's ratio ν, shear modulus G, and mass density ρ. It is assumed that the material properties of the beam vary continuously in the thickness direction (Z) according to a power law distribution as follows [7, 8]:

$$P(z) = (P_t - P_b)\left(\frac{z}{h} + \frac{1}{2}\right)^k + P_b \tag{1}$$

where P_t and P_b are respectively the material properties at the top and bottom surfaces of the FGB, k dictates the material variation profile through the thickness of the beam and is a non-negative parameter. Three special cases are observed in the above equation. The linear variation of the composition of the top and bottom surfaces of the FGB is represented by $k = 1$, $k = 0$ indicates the FGB made of full material of the top surface whereas $k = \infty$ represents the FGB made of full material of the bottom surface.

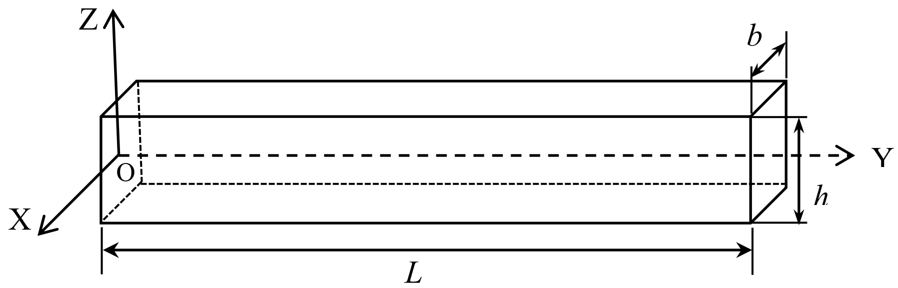

Figure 1: The co-ordinate system and notation for a FGB

Displacements v_1 and w_1 along the Y and Z directions of a point on the cross-section are given by

$$v_1(y,z,t) = v(y,t) - z\frac{\partial w(y,t)}{\partial y} + \varphi(z)\psi(y,t) \tag{2}$$

$$w_1(y,z,t) = w(y,t) \tag{3}$$

where v and w are the corresponding displacements of the point on the neutral axis. $\varphi(z)$ is the shape function which characterises the distribution of the transverse shear stress through the thickness of the beam and can be ascertained using different beam theories. In the current investigation, one of the higher order deformation beam theories, the parabolic shear deformation beam theory [9, 10] is used which assumes:

$$\varphi(z) = z(1 - \frac{4z^2}{3h^2}) = z(1 - az^2) \tag{4}$$

where $a = \dfrac{4}{3h^2}$. The transverse shear strain $\psi(z)$ at any point on the neutral axis can be expressed as

$$\psi(y,t) = w' + \phi(y,t) \tag{5}$$

where a prime represents differentiation with respect to space *y*. ϕ is the total bending rotation of the cross-sections at any point on the neutral axis which is taken to be an unknown function.

Thus displacement v_1 in Equation (2) can be rewritten as:

$$v_1 = v + z[(1 - az^2)\phi - az^2 w'] \tag{6}$$

The normal and shear strains in the usual notation are:

$$\varepsilon_{yy} = \frac{\partial v_1}{\partial y} = v' + z\left[\left(1 - az^2\right)\phi' - az^2 w''\right] \tag{7}$$

$$\gamma_{yz} = \frac{\partial v_1}{\partial z} + \frac{\partial w_1}{\partial y} = \left(1 - 3az^2\right)\left(w' + \phi\right) \tag{8}$$

Assuming that the material of FGB obeys Hooke's law, the stresses in the beam can be expressed as:

$$\sigma_{yy} = E(z)\varepsilon_{yy}, \ \tau_{yz} = G(z)\gamma_{yz} \tag{9}$$

The potential energy of the FGB is given in the usual notation as:

$$\begin{aligned} U &= \frac{1}{2}\int(\sigma_{yy}\varepsilon_{yy} + \tau_{yz}\gamma_{yz})dV \\ &= \frac{1}{2}\int_0^L \left\{A_0 v'^2 - 2A_3 v' w'' + A_2 w''^2 + 2A_1 v'\phi' - 2A_4 w''\phi' + A_5\phi'^2\right\}dy \\ &\quad + \frac{1}{2}\int_0^L A_6\left\{w'^2 + 2w'\phi + \phi^2\right\}dy \end{aligned} \tag{10}$$

The kinetic energy of the FGB is:

$$
\begin{aligned}
T &= \frac{1}{2}\int_0^L \left\{ \int \rho(z)(\dot{v}_1^2 + \dot{w}_1^2) \right\} dA\,dy \\
&= \frac{1}{2}\int_0^L \left\{ I_0(\dot{v}^2 + \dot{w}^2) - 2I_3 \dot{v}\dot{w}' + 2I_1 \dot{v}\dot{\phi} + I_5 \dot{\phi}^2 - 2I_4 \dot{w}'\dot{\phi} + I_2 \dot{w}'^2 \right\} dy
\end{aligned}
\tag{11}
$$

where an over dot represents differentiation with respect to time t. Parameters A_i and I_i are defined as:

$$
\left.
\begin{array}{ll}
\{b_0, b_1, b_3\} = \int \rho(z)\{1, z, az^3\} dA & \\
\{a_0, a_1, a_3\} = \int E(z)\{1, z, az^3\} dA & \\
A_6 = \int G(z)(1-3az^2)^2\, dA & \\
A_0 = a_0 & I_0 = b_0 \\
A_1 = a_1 - a_3 & I_1 = b_1 - b_3 \\
A_2 = a_3^2 & I_2 = b_3^2 \\
A_3 = a_3 & I_3 = b_3 \\
A_4 = (a_1 - a_3)a_3 & I_4 = (b_1 - b_3)b_3 \\
A_5 = (a_1 - a_3)^2 & I_5 = (b_1 - b_3)^2
\end{array}
\right\}
\tag{12}
$$

Hamilton's principle states

$$\delta \int_{t_1}^{t_2} (T - U)\, dt = 0 \tag{13}$$

where t_1 and t_2 are the time intervals in the dynamic trajectory, and δ is the usual variational operator.

Substituting potential (U) and kinetic (T) energies from Equations (10) and (11) into Hamilton's principle in Equation (13), using the δ operator, integrating each term by parts, and then collecting terms yield the governing differential equations and natural boundary conditions in free vibration of the FGB. The entire procedure has been executed using the application of symbolic computation [11]. The governing differential equations are obtained as,

$$-I_0\ddot{v} + A_0 v'' + I_3\ddot{w}' - A_3 w''' - I_1\ddot{\phi} + A_1\phi'' = 0 \tag{14}$$

$$-I_3\ddot{v}' + A_3 v''' + I_2\ddot{w}'' - I_0\ddot{w} + A_6 w'' - A_6 w'''' - I_4\ddot{\phi}' + A_4\phi''' + A_6\phi' = 0 \tag{15}$$

$$A_1 v'' - I_1\ddot{v} - A_6 w' + I_4\ddot{w}' + A_4 w''' - I_5\ddot{\phi} + A_5\phi'' - A_6\phi = 0 \tag{16}$$

As a by-product of the Hamiltonian formulation, the natural boundary conditions are also obtained analytically for axial force, shear force, bending moment and higher order moment as,

$$F = -A_0 v' + A_3 w'' - A_1 \phi' \tag{17}$$

$$S = I_3 \ddot{v} - A_3 v'' - I_2 \ddot{w}' - A_6 w' - A_2 w''' + I_4 \ddot{\phi} - A_4 \phi'' - A_6 \phi \tag{18}$$

$$M = -A_1 v' + A_4 w'' - A_5 \phi' \tag{19}$$

$$M_h = A_3 v' - A_2 w'' + A_4 \phi' \tag{20}$$

Assuming harmonic oscillation so that

$$v(y,t) = V(y)e^{i\omega t}, \; w(y,t) = W(y)e^{i\omega t}, \; \phi(y,t) = \Phi(y)e^{i\omega t} \tag{21}$$

where $V(y)$, $W(y)$ and $\Phi(y)$ are amplitudes of v, w and ϕ, and ω is angular or circular frequency. Introducing the differential operator $D = d/d\xi$ and the non-dimensional length ξ as:

$$\xi = y/L \tag{22}$$

The differential equations of motion in Equations (14-16) can be transformed into following forms:

$$\left.\begin{aligned} & c_1 V(\xi) - c_2 W(\xi) + c_3 \Phi(\xi) = 0 \\ & c_2 L V(\xi) + c_4 W(\xi) + c_5 L \Phi(\xi) = 0 \\ & c_3 V(\xi) - c_5 W(\xi) + c_6 \Phi(\xi) = 0 \end{aligned}\right\} \tag{23}$$

where

$$\left.\begin{aligned} & c_1 = I_0\omega^2 L^3 + A_0 L D^2 \quad c_2 = I_3\omega^2 L^2 D + A_3 D^3 \quad c_3 = I_1\omega^2 L^3 + A_1 L D^2 \\ & c_4 = I_0\omega^2 L^4 - I_2\omega^2 L^2 D^2 - A_2 D^4 + A_6 L^2 D^2 \\ & c_5 = I_4\omega^2 L^3 D + A_4 L D^3 + A_6 L^3 D \\ & c_6 = I_5\omega^2 L^3 - A_6 L^3 + A_5 L D^2 \end{aligned}\right\} \tag{24}$$

The three equations in Equation (23) can be combined into one 8th order ordinary differential equation, which satisfies each of $V(\xi)$, $W(\xi)$ and $\Phi(\xi)$ as follows,

$$(D^8 + d_1 D^6 + d_2 D^4 + d_3 D^2 + d_4)H = 0 \tag{25}$$

where

$$H = V(\xi) \text{ or } W(\xi) \text{ or } \Phi(\xi) \tag{26}$$

The characteristic or auxiliary equation of the differential equation (25) can be reduced to a quartic equation which can be solved using standard procedure [13]. Then by taking the square root of the four roots (which could be real or complex) of the quartic, the eight roots r_j $(j=1,2,\ldots,8)$ of the characteristic equation can be computed. Thus the solutions for $V(\xi)$, $W(\xi)$ and $\Phi(\xi)$ are given by

$$V(\xi)=\sum_{j=1}^{8}P_j e^{r_j\xi},\ W(\xi)=\sum_{j=1}^{8}Q_j e^{r_j\xi},\ \Phi(\xi)=\sum_{j=1}^{8}R_j e^{r_j\xi} \tag{27}$$

where P_j, Q_j and R_j $(j=1,2,\ldots,8)$ are three different sets of eight constants.

The derivative of the transverse displacement, which is considered to be one of the degrees of freedom resulting from the higher order shear deformation, can be obtained as:

$$W'(\xi)=\sum_{j=1}^{8}Q_j r_j e^{r_j\xi} \tag{28}$$

Three sets of eight constants in Equation (27) can be defined as vectors **P**, **Q** and **R** as follows:

$$\mathbf{P}=\begin{bmatrix}P_1\\P_2\\\vdots\\P_8\end{bmatrix},\quad \mathbf{Q}=\begin{bmatrix}Q_1\\Q_2\\\vdots\\Q_8\end{bmatrix},\quad \mathbf{R}=\begin{bmatrix}R_1\\R_2\\\vdots\\R_8\end{bmatrix} \tag{29}$$

The three sets of constants are not all independent and can be related to each other using Equation (23). The choice of relating two sets of the constants in terms of the third one is arbitrary. In the current investigation, P_j is chosen to be the base set of constants to relate to R_j and Q_j as follows:

$$Q_j=\alpha_j P_j,\qquad R_j=\beta_j P_j \tag{30}$$

where

$$\alpha_j=\frac{L(c_1c_5-c_2c_3)}{c_2c_5L+c_3c_4},\qquad \beta_j=-\frac{c_1c_4+c_2^2L}{c_3c_4+c_2c_5} \tag{31}$$

Similarly the amplitudes of the axial, shear forces, bending moment and higher order bending moment are obtained in terms of P_j with the help of Equation (23) in non-dimensional form as

$$F = -\frac{1}{L^2}(LA_0V' + A_3W'' + LA_1\Phi') = -\frac{1}{L^2}\sum_{j=1}^{8} (A_0L + A_3\alpha_j r_j + A_1\beta_j L) r_j e^{r_j\xi} P_j \tag{32}$$

$$\begin{aligned} S &= \frac{1}{L^3}(I_3\omega^2 L^3 V - LA_3V'' + I_2\omega^2 L^2 W' - A_6 L^2 W' + A_2 W''' - (I_4\omega^2 + A_6)L^3\Phi - LA_4\Phi'' \\ &= \frac{1}{L^3}\sum_{j=1}^{8} \begin{bmatrix} (I_3\omega^2 L^3 + LA_3 r_j^2) + (I_2\omega^2 L^2 - A_6L^2 + A_2 r_j^2)\alpha_j r_j \\ -(I_4\omega^2 + A_6)L^3\beta_j - LA_4\beta_j r_j^2 \end{bmatrix} e^{r_j\xi} P_j \end{aligned} \tag{33}$$

$$M = \frac{1}{L^2}(-A_1LV' + A_4W'' - A_5L\Phi') = \frac{1}{L^2}\sum_{j=1}^{8} (-A_1L + A_4\alpha_j r_j - A_5L\beta_j) r_j e^{r_j\xi} P_j \tag{34}$$

$$M_h = \frac{1}{L^2}(A_3LV' - A_2W'' + A_4L\Phi') = \frac{1}{L^2}\sum_{j=1}^{8} (A_3L - A_2\alpha_j r_j + A_4L\beta_j) r_j e^{r_j\xi} P_j \tag{35}$$

2.2 Dynamic stiffness formulation

The dynamic stiffness matrix of the FGB can be formulated by applying boundary conditions for displacements and forces at the ends of the beam. Figure 2 shows the sign convention for axial force, shear force and bending moment used in this paper when applying for the boundary conditions.

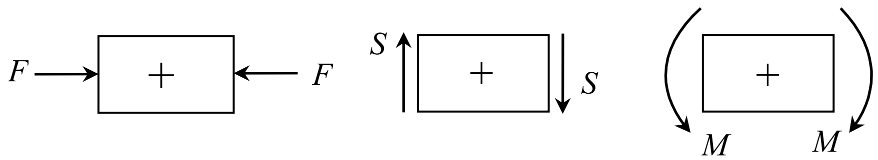

Figure 2: Sign convention for positive axial force, shear force and bending moment.

The boundary conditions for the displacements at the ends of the FGB are,

$$\left.\begin{aligned} &y = 0\ (\xi = 0):\ V = V_1, W = W_1, \Phi = \Phi_1, W' = W_1' \\ &y = L\ (\xi = 1):\ V = V_2, W = W_2, \Phi = \Phi_2, W' = W_2' \end{aligned}\right\} \tag{36}$$

The boundary conditions for the forces at the both ends of the FGB are,

$$\left.\begin{aligned} &y = 0\ (\xi = 0):\ F = F_1, S = S_1, M = M_1, M_h = M_{h1} \\ &y = L\ (\xi = 1):\ F = -F_2, S = -S_2, M = -M_2, M_h = M_{h2} \end{aligned}\right\} \tag{37}$$

Therefore, the vectors of the displacement and force are defined as:

$$\left.\begin{aligned} \boldsymbol{\delta} &= \begin{bmatrix} V_1 & W_1 & \Phi_1 & W_1' & V_2 & W_2 & \Phi_2 & W_2' \end{bmatrix}^T \\ \mathbf{F} &= \begin{bmatrix} F_1 & S_1 & M_1 & M_{h1} & F_2 & S_2 & M_2 & M_{h2} \end{bmatrix}^T \end{aligned}\right\} \tag{38}$$

with superscript T denoting a transpose.

Substituting the boundary conditions for displacement of Equations (36) into Equations (27-28), the relationship between the displacement vector $\boldsymbol{\delta}$ and the constant vector $\mathbf{P}$ is obtained as

$$\boldsymbol{\delta} = \mathbf{B}\,\mathbf{P} \tag{39}$$

where

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 \\ r_1\alpha_1 & r_2\alpha_2 & r_3\alpha_3 & r_4\alpha_4 & r_5\alpha_5 & r_6\alpha_6 & r_7\alpha_7 & r_8\alpha_8 \\ e^{r_1} & e^{r_2} & e^{r_3} & e^{r_4} & e^{r_5} & e^{r_6} & e^{r_7} & e^{r_8} \\ \alpha_1 e^{r_1} & \beta_2 e^{r_2} & \beta_3 e^{r_3} & \beta_4 e^{r_4} & \beta_5 e^{r_5} & \beta_6 e^{r_6} & \beta_7 e^{r_7} & \beta_8 e^{r_8} \\ \beta_1 e^{r_1} & \beta_2 e^{r_2} & \beta_3 e^{r_3} & \beta_4 e^{r_4} & \beta_5 e^{r_5} & \beta_6 e^{r_6} & \beta_7 e^{r_7} & \beta_8 e^{r_8} \\ r_1\alpha_1 e^{r_1} & r_2\alpha_2 e^{r_2} & r_3\alpha_3 e^{r_3} & r_4\alpha_4 e^{r_4} & r_5\alpha_5 e^{r_5} & r_6\alpha_6 e^{r_6} & r_7\alpha_7 e^{r_7} & r_8\alpha_8 e^{r_8} \end{bmatrix} \tag{40}$$

Similarly, by substituting the boundary conditions for forces given in Equations (37) into Equations (32 - 35), the relationship between the force vector $\mathbf{F}$ and the constant vector $\mathbf{P}$ is obtained as

$$\mathbf{F} = \mathbf{A}\,\mathbf{P} \tag{41}$$

By eliminating the constant vector $\mathbf{P}$ from Equations (39) and (41), the force vector $\mathbf{F}$ and the displacement vector $\boldsymbol{\delta}$ can be related to give the dynamic stiffness matrix relationship of the beam as

$$\mathbf{F} = \mathbf{K}\,\boldsymbol{\delta} \tag{42}$$

where

$$\mathbf{K} = \mathbf{A}\,\mathbf{B}^{-1} \tag{43}$$

is the required frequency dependent 8×8 dynamic stiffness matrix of the beam.

The above dynamic stiffness matrix **K** can be used to compute the natural frequencies and mode shapes of either an individual FGB or an assembly of FGBs for different boundary conditions. A reliable and accurate method of computing the natural frequencies using the dynamic stiffness method is to apply the well-established Wittrick-Williams algorithm [12] which is ideally suited to solve transcendental eigenvalue problems such as the one in this paper.

The Wittrick and Williams algorithm uses the Sturm sequence property of the dynamic stiffness matrix and has featured in literally hundreds of papers. It ensures that no natural frequencies of the structure being analysed are missed. Clearly, this is not possible in the conventional finite element or other approximate methods. For a detailed insight of the algorithm, interested readers are referred to the original work of Wittrick and Williams [12].

3 Conclusions

A systematic procedure to derive the dynamic stiffness properties of a functionally graded beam using a higher order shear deformation theory is presented in this paper. The governing differential equations of motions have been derived through the application of Hamiltonian mechanics and by appropriate utilisation of symbolic computation. The frequency dependent dynamic stiffness matrix has been developed for further applications in free vibration and response analyses. Further investigation is underway and the validation of the proposed theory and illustration of the method with detailed numerical results will be reported at a later stage.

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