Asset Allocation with Multiple Analysts' Views: A Robust Approach

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Abstract

Retail investors often make decisions based on professional analysts' investment recommendations. Although these recommendations contain up-to-date financial information, they are usually expressed in sophisticated but vague forms. In addition, the quality differs from analyst to analyst and recommendations may even be mutually conflicting. This paper addresses these issues by extending the Black-Litterman (BL) method, and developing a multi-analyst portfolio selection method, balanced against any over-optimistic forecasts. Our methods accommodate analysts' ambiguous investment recommendations and the heterogeneity of data from disparate sources. We prove the validity of our model, using an empirical analysis of around 1000 daily financial newsletters collected from two top-10 Taiwanese brokerage firms over a two-year period. We conclude that analysts' views contribute to the investment allocation process and enhance the portfolio performance. We confirm that the degree of investors' confidence in these views influences the portfolio outcome, thus extending the idea of the BL model and improving the practicality of robust optimisation.

Keywords: analysts' recommendation; Black-Litterman model; fuzzy logic; portfolio selection; robust optimisation.

JEL classification: G11

1. Introduction

Traditional portfolio selection theory (Markowitz, 1952) maximises anticipated returns based on a given level of risk. Input parameters (anticipated return vector and corresponding (co-)variances (Markowitz, 1952)) are computed using only market returns data, meaning that important financial information updates e.g. firm-specific earnings announcements, cannot be used to support portfolio selection. To improve this, the BL method (Black and Litterman, 1991; Black and Litterman, 1992) consists of both the market model and the view model. The BL's market model specifies a normal distribution for the stock return vector with a prior distribution of the expected asset returns elicited from the equilibrium returns (Meucci, 2010; Schöttle et al., 2010). The BL's view model includes a fund manager's current views on financial assets. The BL method has been extended to investigate a wide range of asset allocation problems. For example, Bertsimas et al. (2012) consider inverse optimisation for the BL method. Fernandes et al. (2012) combine the re-sampling method with the BL method. O'Toole (2017) investigates an alternative derivation of the BL model that offers an efficient approach to target active risk, while van der Schans and Steehouwer (2017) propose a time-dependent BL approach.

The asset allocation methods are commonly used, but their inputs e.g. return in the mean-variance model and fund manager's view in the BL method, are subject to uncertainty and ambiguity (Kaya, 2017; O'Toole, 2017; de Jong, 2018). To reduce their impact, the input parameters must be as accurate as possible. This paper discusses retail investors' portfolio selection with multiple professional analysts' recommendations, in three aspects. First, although financial analysts' investment recommendations contain up-to-date information, they are usually vague. We will use the fuzzy set theory to quantify ambiguous forecasts and apply them to build the view model. Secondly, we extend the model from a single fund manager's view to one that deals with multiple analysts' views. These views could be contradictory, which we address by following the multi-expert approach of Lutgens and Schotman (2010) and undertaking a worst-case scenario analysis. This approach counters over-optimistic opinions and alleviates the associated risk in financial investment. Finally, we take a robust counterpart approach with uncertainty sets of different confidence levels to tackle the sampling error problem and associated heterogeneity of data originating from disparate sources i.e. historical market data and analysts' investment forecasts. This gives an alternative way to the Bayesian approach adopted in the BL method.

Our first research strand in this paper is the fuzzy set theory developed by Zadeh (1965). This provides a solution to the problems of uncertainties, imprecision, and contradictions found in human verbal expressions, and enables us to interpret ambiguous forecasts by using fuzzy variables. We adopt the possibilistic approach of Carlsson et al. (2002), which uses trapezoidal fuzzy variables for asset returns and selects portfolios with the highest utility score. Other approaches (Bartkowiak and Rutkowska, 2017) use fuzzy random variables defined in Puri and Ralescu (1986) to process experts' linguistic views.

Our second strand of research is the robust counterpart approach. Recent studies reveal that parameter estimates based on historical market data are subject to sampling errors (Chopra and Ziemba, 1993; Schöttle and Werner, 2009; Fernandes et al., 2012; de Jong, 2018). The robust counterpart approach is a useful alternative because it includes a wide range of possible input parameter values without complicated adjustment to the original optimisation framework (Ben-Tal and Nemirovski, 1998; Gregory et al., 2011). Robust counterpart optimisation has also long been recognised to be closely related to the Bayesian approach (Fabozzi et al., 2007, 2010), which forms the main motivation of this paper.

To build a final view model is always challenging, because there are so many sources of potentially conflicting information. In the multi-criteria decision-making literature, a widely used practice for combining multiple experts' opinions is the weighted sum model (WSM) (see e.g. Triantaphyllou, 2000). The well-known Delphi technique for gathering and processing options from multiple experts, summarises the views using the mean or median (Hsu and Sandford, 2007). The BL model, as a Bayesian method, follows the standard approach to determine the weights of the WSM when pooling multiple experts' views; it assumes that the variances for individual experts' views are available (see, e.g. Bartkowiak and Rutkowska, 2017), and uses their reciprocals as the corresponding weights. However, without relevant knowledge, it is still a difficult task for retail investors to find these variance parameters. We follow the approach of Lutgens and Schotman (2010) to determine the weights in the WSM; their approach avoids the bias of potentially over-optimistic analysts by investigating a worst-case scenario analysis. In summary, we extend the BL method through realistic inclusion and evaluation of multiple financial analysts' recommendations in a portfolio selection problem.

We assess the effectiveness of the developed method through a large-scale case study on the Taiwanese stock market with 148 capitalized stocks, representing nearly 90% of the market over a two-year period. Nearly 1000 daily financial newsletters were collected from two top-10 Taiwanese financial brokerage firms, to help extract stock forecast recommendations. This is the first practical application of the robust portfolio selection model that uses professional analysts'[1] data to evaluate the value of financial recommendations in portfolio selection.

In Section 2, we develop a multi-analyst method to incorporate multiple information sources into the portfolio decision-making process. We adopt multiple uncertainty sets to enhance the framework and construct our robust counterpart approach. Section 3 provides an empirical case study to analyse the Taiwanese stock market. Section 4 summarises the findings and concludes this paper. Propositions and proofs are in the Appendix.

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[1] Others in the literatures did not appear to have significantly used analysts' data. For example, Huang et al. (2010) consider four experts, each using one sub-sample of the market data to form their prior of the portfolio returns. Bartkowiak and Rutkowska (2017) create a couple of linguistic views (see Table 1 in their paper), but with no clear/direct references to data of real-life professional analysts' recommendations.

2. Portfolio selection with multiple analysts' recommendations

This section summarises the mean-variance model and the BL method and builds on this to develop a new approach.

2.1. *The mean-variance model and Black-Litterman method*

Consider an investor constructing a portfolio with n risky assets over a single-period horizon. Markowitz (1952) formulates the portfolio selection problem as maximising the anticipated return of the portfolio, subject to a given risk characterised by its variance. The anticipated portfolio return in Markowitz (1952) has been interpreted as the expected return (see, e.g., Fabozzi et al., 2010), and the portfolio selection problem is usually formulated to maximise the portfolio return based on variance as a risk measure, i.e. optimising the following mean-variance utility function:

$$(P_{MV}) \qquad R = \mu^T x - \frac{\lambda}{2} x^T \Sigma x \tag{1}$$

where $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$ is the vector of the portfolio weights, and λ the risk aversion coefficient. $r = [r_1, \dots, r_n]^T$ is the vector of the asset returns with mean vector $\mu = [\mu_1, \dots, \mu_n]^T$ and the covariance matrix $\Sigma = [\sigma_{ij}] \in \mathbb{R}^n \times \mathbb{R}^n$. The optimal portfolio x^* can be obtained by maximising the mean-variance utility in equation (1):

$$x^* = \frac{1}{\lambda} \Sigma^{-1} \mu. \tag{2}$$

The parameters in the vector μ and the covariance matrix Σ can be estimated with historical market data in a number of ways. For example, the maximum likelihood estimates can be used (see, e.g., Becker et al., 2015). Throughout the paper, they are denoted as

$$\hat{\mu} = [\hat{\mu}_1, \dots, \hat{\mu}_n]^T \qquad \text{and} \qquad \hat{\Sigma} = [\hat{\sigma}_{ij}]_{n \times n}. \tag{3}$$

Black and Litterman (1991, 1992) propose a method to construct a portfolio consisting of the market and the view models, where a fund manager can base upon, with views on updated financial information, to construct an investment portfolio. Specifically, the market model in the BL method assumes that the return vector follows a normal distribution:

$$r \sim N(\mu, \Sigma). \tag{4}$$

As a Bayesian approach, a prior distribution of the vector μ in (1) is specified in the BL method by invoking an equilibrium argument, $\mu \sim N(\pi, \rho\Sigma)$, where π represents the best guess for μ and $\rho\Sigma$ characterises the uncertainty of this guess (Meucci, 2010; Schöttle et al., 2010).

In addition, the view model in the BL method is specified by a fund manager to reflect his/her views on the future returns of the assets as follows:

$$P\mu \sim N(v, \Omega), \tag{5}$$

where the hyper-parameter vector $v = [v_1, \dots, v_n]^T$ and covariance matrix $\Omega = [\tau_{ij}]$ quantify the average returns and the corresponding uncertainties of the views respectively. The $m \times n$ matrix P is termed a stock-pick matrix, describing the stocks which the fund manager provides his/her views. m represents the number of views.

If the manager has a view about each of the m assets indexed by $i_1, \dots, i_m$, then the matrix $P = [p_{ij}]$ is an $m \times n$ matrix of 0 and 1, where the elements in the ith ($i = 1, \dots, m$) row are all equal to 0 except for the entry in column i_j that takes the value of one, i.e.

$$p_{ij} = \begin{cases} 1 & if\ j = i_j \\ 0 & otherwise \end{cases}.$$

The view model (5) can then be written as $\mu_j \sim N(v_j, \tau_{jj})$ for $j = i_1, \dots, i_m$. Consider an example where in total there are 4 stocks with

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here, equation (5) indicates that the fund manager has views about the second stock and the fourth stock respectively, as the second entry of the first row and the fourth entry of the second row in the pick matrix P are ones, and all the other entries are zeros.

Based on the prior distribution (4) and the view model (5), the posterior distribution of the return can be derived using the Bayes' rule. The obtained posterior mean vector and covariance matrix from the BL method can be used to replace the input parameters in the mean-variance utility (1), and to find the portfolio by maximising the mean-variance utility in problem (1) with the posterior means and covariance matrix (O'Toole, 2017).

2.2. *Multi-analyst portfolio selection with fuzzy logic*

The BL method does not specify any method of eliciting a fund manager's views in a quantitative manner. In addition, it assumes that all views come from a single individual, e.g. the fund manager, which may not apply to retail investors who may consult multiple financial analysts. This section extends the BL method by: (a) investigating how the view of a professional analyst presented in a vague and ambiguous format can be quantified in the form of equation (5); and (b) considering the rival and potentially conflicting information problem when pooling the views from multiple financial analysts for portfolio selection.

2.2.1 *Quantifying analysts' views with fuzzy logic*

We consider a problem where an investor consults Z professional analysts for investment forecasts/recommendations. Let $\mathbb{Z}$ denote the set of the financial analysts and S_z denote the set of stocks for which an analyst $z \in \mathbb{Z}$ has made recommendations. We assume that there are m_z ($m_z < n$) stocks picked by analyst z. In this sub-section, we focus on just one analyst $z \in \mathbb{Z}$.

Real-life analysts present their investment recommendations differently and there is no standard style and format. The fuzzy set theory deals with this by adopting a standardized format so that these recommendations can be quantified and efficiently included in the portfolio computation. We adopt the fuzzy set theory approach and follow Carlsson et al. (2001), Gupta et al. (2008) and Bartkowiak and Rutkowska (2017), when dealing with linguistic views/recommendations. More specifically, we consider analysts' investment forecasts as qualitative data and assume the expected return μ_i of each stock i (for $i \in S_z$) elicited from analyst $z \in \mathbb{Z}$ is a trapezoidal fuzzy variable characterized by a quadruplet $(\mu_{zi}^{m-}, \mu_{\text{zi}}^{m+}, \sigma_{\text{zi}}^{-}, \sigma_{zi}^{+})$ with the following membership function:

$$M_z(\mu_i) = \begin{cases} 1 & \mu_i \in [\mu_{zi}^{m_-}, \mu_{zi}^{m_+}] \\ 1 - \frac{\mu_{zi}^{m_-} - \mu_i}{\sigma_{zi}^-} & \mu_i \in [\mu_{zi}^{m_-} - \sigma_{zi}^-, \mu_{zi}^{m_-}] \\ 1 - \frac{\mu_i - \mu_{zi}^{m_+}}{\sigma_{zi}^+} & \mu_i \in [\mu_{zi}^{m_+}, \mu_{zi}^{m_+} + \sigma_{zi}^+] \\ 0 & otherwise \end{cases} \tag{6}$$

where $[\mu_{zi}^{m_-}, \mu_{zi}^{m_+}]$ denotes the return tolerance interval, and σ_{zi}^- and σ_{zi}^+ represent the left width and right width of the membership function, respectively. The membership function (6) is suitable for financial investment forecasts/recommendations in our empirical analysis. Figure. 1 illustrates the above membership function.

[Insert Figure 1 here]

To extract the investment information from a fuzzy-set membership function, this paper follows the method of crisp possibilistic interpretation in Carlsson and Fuller (2001), and considers the crisp possibilistic mean and variance of each asset $i \in S_z$:

$$\tilde{\mu}_{zi} \coloneqq E(\mu_i) = \frac{\mu_{zi}^{m_-} + \mu_{zi}^{m_+}}{2} + \frac{\sigma_{zi}^+ - \sigma_{zi}^-}{6} \tag{7}$$

$$\tilde{\sigma}_{zi}^2 \coloneqq Var(\mu_i) = \left[\frac{\mu_{zi}^{m_+} - \mu_{zi}^{m_-}}{2} + \frac{\sigma_{zi}^- + \sigma_{zi}^+}{6}\right]^2 + \frac{(\sigma_{zi}^- + \sigma_{zi}^+)^2}{72}. \tag{8}$$

Hence, the view model (5) can approximately be written as:

$$\mu_i \sim N(\tilde{\mu}_{zi}, \tilde{\sigma}_{zi}^2) \qquad \text{for } i \in S_z. \tag{9}$$

To pool the view model (9) with the market model (4), a widely used method is the Bayesian approach (as in the BL method) or the Stein's "shrinkage" estimator (see, e.g., Becker et al., 2015):

$$\check{\mu}_{zi}(s_{zi}) = (1 - s_{zi})\hat{\mu}_i + s_{zi}\tilde{\mu}_{zi},$$

where s_{zi} ($0 \leq s_{zi} \leq 1$) are the corresponding weights.

This paper assumes that the analysts make forecasts for the assets only if they disagree with their historical performances, i.e. the corresponding estimates obtained from the historical data are regarded to be out-of-date; they are replaced with the analysts' recommendations. Hence, we choose

$$s_{zi} = \begin{cases} 0 & i \notin S_z \\ 1 & i \in S_z \end{cases}.$$

For the assets without analysts' recommendations, the historical asset performances are adopted to obtain parameter estimates. Hence, we define the return vector $\check{\mu}_z = [\check{\mu}_{z1}, \check{\mu}_{z2}, \dots, \check{\mu}_{zn}]^T$ for each analyst $z \in \mathbb{Z}$ as follows:

$$\check{\mu}_{zi} = \begin{cases} \hat{\mu}_i & i \notin S_z \\ \tilde{\mu}_{zi} & i \in S_z \end{cases} \tag{10}$$

where $\hat{\mu}_i$ and $\tilde{\mu}_{zi}$ are given by (3) and (7) respectively. Likewise, we define

$$\check{\sigma}_{zi}^2 = \begin{cases} \hat{\sigma}_{ii} & i \notin S_z \\ \tilde{\sigma}_{zi}^2 & i \in S_z \end{cases} \tag{11}$$

where $\hat{\sigma}_{ii}$ and $\tilde{\sigma}_{zi}^2$ are given by (3) and (8) respectively.

Professional analysts rarely provide information about the relationships between assets; hence, it is difficult to elicit the entire covariance structure of the asset returns using the financial analysts' forecasts. We incorporate a hybrid approach to addressing this issue: the correlation coefficients ρ_{ij} $(i, j = 1, \dots, n)$ are estimated using historical market data, and the covariance matrix $\check{\Sigma}_z$ is constructed with the correlation coefficients ρ_{ij} and the individual variances in (11):

$$\check{\Sigma}_z := \left[\rho_{ij}\check{\sigma}_{zi}\check{\sigma}_{zj}\right]_{n\times n}. \tag{12}$$

After the view model (9) is elicited using the fuzzy logic, we re-formulate the objective function in equation (1). We follow Watada (1997) and Gupta et al. (2008) to express the ambiguous aspiration level of the utility $\tilde{R}_z = \check{\mu}_z{}^T x - \frac{\lambda}{2} x^T \check{\Sigma}_z x$ by incorporating a nonlinear logistic membership function:

$$M_{\tilde{R}_z}(x) = \frac{1}{1+\exp(-\theta_z(\tilde{R}_z - R^{Target}))}, \tag{13}$$

where R^{Target} is the benchmark set by the investor as the aspiration level for the portfolio. Parameter θ_z $(0 < \theta_z < \infty)$ denotes the credibility level about analyst $z \in \mathbb{Z}$. The value of credibility θ_z can be chosen empirically by the investor based on prior knowledge to reflect his/her preference. Fig. 2 illustrates the membership function for different credibility levels of θ_z.

[Insert Figure 2 here]

2.2.2. Elicitation of the input parameters from analysts' investment recommendations

This section discusses how investment recommendations are presented by analysts in the daily financial newsletters in our study. The daily financial newsletters consist of analysts' views regarding selected stocks from the market. Overall, the ways that the analysts express their views differ from time to time and from stock to stock, resulting in different presentation formats. This may be attributable to their confidence levels in the accuracy of their forecasts or simply a matter of preferable styles of presentation. Here we consider three popular formats used in the analysts' financial newsletters.

Case (I): Recommendations expressed as resistance and support

Some investment recommendations are presented in the form of a given price resistance and price support, as shown in Table 1. This format doesn't explicitly indicate an investment recommendation, e.g. buy and sell, and is opened to investors for interpretations.

To convert such recommendations into a fuzzy variable, we compute the *support* and *resistance* using the stock's closing price *(CP)*, as well as the price support *(PS)* and price resistance *(PR)* provided by the analyst ($PR \geq PS$), i.e. $support = \frac{PS-CP}{CP}$ and $resistance = \frac{PR-CP}{CP}$. We use the average of the support and resistance to obtain the lower limit of the tolerance interval , i.e. $m_- = \frac{resistance+support}{2}$, with the left deviation equal to $\sigma_- = m_- - support = \frac{resistance-support}{2}$. Therefore, the fuzzy variable of this recommendation type is characterized by a quadruplet form:

$$(m_-, m_+, \sigma_-, \sigma_+) = \left(\frac{resistance+support}{2}, resistance, \frac{resistance-support}{2}, 0\right).$$

For example, Table 1 suggests a stock forecast implying a potential buying action on XXXX Technology with $support = \frac{89-93.3}{93.3} = -4.61\%$ and $resistance = \frac{100-93.3}{93.3} = 7.18\%$. Hence, $m_- = \frac{-4.61\%+7.18\%}{2} = 1.285\%$ and $\sigma_- = 1.285\% - (-4.61\%) = 5.895\%$. Therefore, the fuzzy interpretation of the recommendation for XXXX Technology is of a quadruplet form $(1.285\%, 7.18\%, 5.895\%, 0\%)$.

[Insert Table 1 here]

Case (II): Recommendations expressed as target price and potential rate

The second type of presentation gives detailed analysis for a stock with a clear investment action. Apart from the closing price *(CP)* of the stock, this form of forecast also provides a target price *(TP)* and a potential rate (PR). Define $TR = (TP - CP)/CP$. The stock forecast for the stock can be expressed in the following quadruplet form $(m_-, m_+, \sigma_-, \sigma_+)$, depending on whether it is a "buy", "neutral", or "sell" recommendation.

Specifically, for "buy" recommendations, we have $PR \geq TR \geq 0$ and the corresponding quadruplet form is taken as $\tilde{\mu}_{Buy}^{Tra} = (m_-, m_+, \sigma_-, \sigma_+) = (TR, PR, TR, 0)$. Similarly, under the condition $PR \geq TR \geq 0$, "neutral" recommendations are interpreted as $\tilde{\mu}_{Neutral}^{Tra} = (m_-, m_+, \sigma_-, \sigma_+) = (TR, TR, TR, PR - TR)$ which is equivalent to a triangular membership function. Finally, for "sell" recommendations, we have $0 \geq TR \geq PR$, and the corresponding quadruplet form is chosen to be $\tilde{\mu}_{Sell}^{Tra} = (m_-, m_+, \sigma_-, \sigma_+) = (PR, TR, 0, 0 - TR)$.

For example, Table 2 shows a sample of stock forecasts with a "neutral" rating, where the fuzzy interpretation is a quadruplet $(1.71\%, 1.71\%, 1.71\%, 0.29\%)$.

[Insert Table 2 here]

Case (III): Recommendations expressed as price boundaries

The third type of presentation provides four price boundaries (PB) for the stock, i.e. $PB_1 < PB_2 < Closing\ Price < PB_3 < PB_4$. These can be converted into four return vertices, i.e. $RV_n =$

$\frac{PB_n - Closing\ Price}{Closing\ Price}$ with $n = 1,2,3,4$ and $RV_1 < RV_2 < RV_3 < RV_4$, in which the corresponding trapezoidal fuzzy variable is $(m_-, m_+, \sigma_-, \sigma_+) = (RV_2, RV_3, RV_2 - RV_1, RV_4 - RV_3)$.

Table 3 gives an example with the fuzzy expressions as $(-2.04\%, 3.50\%, 4.08\%, 2.48\%)$ for XXXX Construction and $(-1.92\%, 1.28\%, 8.33\%, 2.31\%)$ for XXXX International.

[Insert Table 3 here]

This section discusses the various formats of analysts' recommendations in our newsletters and interpret them using the fuzzy approach. Obviously, analysts' forecasts are not without biases and often could be affected by being over-confident or over-optimistic. The portfolio optimisation approach in the following section addresses this issue by using a max-min approach to deal with the rival information sources from different analysts. With this approach, the over-optimistic view will be tackled and eliminated in the stock selection decision making process when constructing investment portfolios. In practice, to reduce any potential systematic biases, retail investors would frequently validate the forecast accuracy by comparing the stock recommendation with its performance afterward.

2.2.3 Portfolio optimisation with multiple analysts' views

In this section, we investigate portfolio selection with multiple analysts' recommendations. We address this by synthesising the different sources of information. In the WSM model (Triantaphyllou, 2000) and the Bayesian approach, the views from multiple experts are synthesised through a weighted average. In this paper, the synthesised views are calculated by averaging the individual mean vectors and covariance matrices:

$$\bar{\mu} = \sum_{z=1}^{Z} \varpi_z \check{\mu}_z \qquad \text{and} \qquad \bar{\Sigma} = \sum_{z=1}^{Z} \varpi_z \check{\Sigma}_z \qquad (14)$$

where $\check{\mu}$ and $\check{\Sigma}_z$ are given in equations (10) and (12). $\varpi_z \geq 0$ (for all $z \in \mathbb{Z}$) are weights to be determined for the WSM model. The weights satisfy the normalization condition $\sum_{z=1}^{Z} \varpi_z = 1$.

To determine the weights in (14), we follow Lutgens and Schotman (2010) and incorporate a max-min approach to deal with the rival information sources from different analysts. Technically, this approach solves the portfolio selection problem in two stages: (a) find the worst-case scenario across the analysts' recommendations, $z \in \mathbb{Z}$; and (b) obtain the optimal portfolio by maximising the objective function with respect to the asset weight vector. This approach protects against the scenario that some analysts' recommendations are over-optimistic, and hence helps to alleviate biases underlying the analysts' recommendations.

Based on the objective function (13), we formulate the portfolio selection problem with multiple analysts' forecasts using the following max-min problem:

$$(F_{MV}) \qquad \max_{x\in\mathbb{R}^n} \min_{z\in\mathbb{Z}} \frac{1}{1+exp\left(-\theta_z\left(\check{\mu}_z{}^T x - \frac{\lambda}{2}x^T \check{\Sigma}_z x - R^{Target}\right)\right)} \, . \tag{15}$$

Proposition 1 in the Appendix shows the solution to problem (15) is given by $x^* = \frac{1}{\lambda}\, {\Sigma^*}^{-1}\mu^*$ with $\Sigma^* = \sum_{z=1}^{Z} \omega_z \check{\Sigma}_z$ and $\mu^* = \sum_{z=1}^{Z} \omega_z \check{\mu}_z$, where ω_z are weights determined by the Lagrange multipliers of (15). Hence, the optimal portfolio of (15) depends on the weighted average of individual mean vectors and covariance matrices of the individual analysts.

We observe that the asset allocation, $x^* = \frac{1}{\lambda}\, {\Sigma^*}^{-1}\mu^*$, has a mathematical form similar to O'Toole (2017); the latter is based on the mean-variance optimisation (1) with mean and (co)variance parameters elicited from the BL model. However, we point out that $\Sigma^* = \sum_{z=1}^{Z} \omega_z \check{\Sigma}_z$ and $\mu^* = \sum_{z=1}^{Z} \omega_z \check{\mu}_z$ in this paper are obtained by synthesising multiple analysts' recommendations, rather than a single fund manager's view in O'Toole (2017). In addition, we note that Bartkowiak and Rutkowska (2017) use a different fuzzy logic approach to elicit experts' views for the BL modelling. To pool various views from experts, they use a Bayesian approach; however, it is not clear in their method how to assign weights to these experts' views. In contrast, this paper addresses the issue by determining the weights ω_z using the max-min approach (15).

2.3. Data heterogeneity: a robust counterpart approach

An important issue of the developed multi-analyst method is the heterogeneity of data from disparate sources, equations (10)-(12). This method uses both the historical market data and multiple analysts' forecasts. The former has sampling errors when market data is used to estimate the expected returns. The latter may suffer from quality differences in the forecasts/recommendations from analyst to analyst.

This section uses the robust-counterpart approach to handle the data heterogeneity problem. This approach is widely used to deal with sampling errors in portfolio management (see, e.g., Ben-Tal and Nemirovski, 1998; Fabozzi et al., 2007). Here we only focus on the expected returns as the fluctuations in the covariance matrix do not significantly influence the optimal solution (Chopra and Ziemba, 1993; Schöttle and Werner, 2009; Ziemba; 2009).

With respect to sampling errors, it is likely that the true values of the expected returns lie within the neighbourhood of the current point estimate, termed uncertainty set. Let $U_z(\check{\mu}_z)$ denote an uncertainty set of the return vector for analyst $z \in \mathbb{Z}$, and $\mu_z \in U_z(\check{\mu}_z)$ as any return parameter vector in this uncertainty set. The uncertainty set $U_z(\check{\mu}_z)$ is usually chosen as a confidence ellipsoid:

$$U_z^{Ellipsoid}(\check{\mu}_z) := \left\{ \mu_z \in \mathbb{R}^n \;\middle|\; (\mu_z - \check{\mu}_z)^T \check{\Sigma}_z^{-1} (\mu_z - \check{\mu}_z) \leq \delta_0^2 \right\}$$
$$= \left\{ \mu_z \in \mathbb{R}^n \;\middle|\; \mu_z = \check{\mu}_z + \delta_0 \check{\Sigma}_z^{\frac{1}{2}} \psi \,, \|\psi\| \leq 1 \right\} \tag{16}$$

where $\check{\mu}_z$ and $\check{\Sigma}_z$ for analyst $z \in \mathbb{Z}$ are given by (10)-(12). The size of the neighbourhood δ_0, also termed robustness level for the uncertainty set, reflects the quality of the estimate $\check{\mu}_z$ as perceived by investors.

Since the true expected return vector can be anywhere in the uncertainty set U_z, we adopt the robust counterpart approach to handle the uncertainty set value[2]. We follow Ben-Tal and Nemirovski (1998) and formulate the robust counterpart approach within the multi-analyst method for portfolio selection

[2] The robust-counterpart approach has attracted a large volume of studies in the recent two decades. See Fabozzi et al. (2007) and Fabozzi et al. (2010) for reference.

$$\max_{x\in\mathbb{R}^n} \min_{z\in\mathbb{Z}} \min_{\mu_z\in U_z} \frac{1}{1+exp\left(-\theta_z\left(\mu_z^T x-\frac{\lambda}{2}x^T\Sigma_z x-R^{Target}\right)\right)} \quad . \tag{17}$$

From a computational perspective, problem (17) can be simplified: Proposition 2 in the Appendix shows that problem (17) can be transformed to a simpler max-min problem below:

$$\left(RF_{MV}^{Joint}\right) \quad \max_{x\in\mathbb{R}^n} \min_{z\in\mathbb{Z}} \frac{1}{1+exp\left(-\theta_z\left(\check{\mu}_z^T x-\frac{\lambda}{2}x^T\Sigma_z x-R^{Target}-\delta_0\left\|\Sigma_z^{\frac{1}{2}}x\right\|\right)\right)} \tag{18}$$

with $\check{\mu}_z - \delta_0 \frac{\Sigma_z x}{\left\|\Sigma_z^{1/2}x\right\|}$ denoting the worst-case scenario of the expected returns.

We next address the issue of disparate data quality between the historical market data and the analysts' forecasts by using different sizes of the uncertainty sets for different information sources. Without loss of generality, we suppose analyst $z \in \mathbb{Z}$ comments on the final m_z $(m_z < n)$ assets. Let the decision vector $x \in \mathbb{R}^n$ be partitioned accordingly with $x = [{x_H}^T, {x_A}^T]^T$, where $x_H \in \mathbb{R}^{n-m_z}$ (or $x_A \in \mathbb{R}^{m_z}$) denotes the vector of the weights associated with the assets whose average returns are estimated using the historical market data (or using the analyst's forecasts). In addition, let

$$\check{\mu}_z = \begin{pmatrix} \check{\mu}_{H_z} \\ \check{\mu}_{A_z} \end{pmatrix} \quad \text{and} \quad \check{\Sigma}_z = \begin{pmatrix} \check{\Sigma}_{HH_z} & \check{\Sigma}_{HA_z} \\ \check{\Sigma}_{AH_z} & \check{\Sigma}_{AA_z} \end{pmatrix}$$

where $\check{\mu}_{H_z}$ and $\check{\mu}_{A_z}$ represent the expected returns from the historical market data (3) and from the analyst z's forecasts (10). Similar partitions are defined for the covariance matrix $\check{\Sigma}_z$.

To deal with the data heterogeneity issue, we assume the investor chooses two different confidence levels, δ_{H_z} and δ_{A_z}, for each analyst $z \in \mathbb{Z}$ and forms two uncertainty sets

$$U_{H_z}^{Ellipsoid}(\check{\mu}_{H_z}) = \left\{ \mu \in \mathbb{R}^{n-m_z} \ \left| (\mu - \check{\mu}_{H_z})^T {\check{\Sigma}_{HH_z}}^{-1} (\mu - \check{\mu}_{H_z}) \le \delta_{H_z}^2 \right.\right\}$$

$$= \left\{ \mu \in \mathbb{R}^{n-m_z} \left| \mu = \check{\mu}_{H_z} + \delta_{H_z} \check{\Sigma}_{HH_z}^{\frac{1}{2}} \psi_{H_z}, \left\|\psi_{H_z}\right\| \le 1 \right.\right\} \tag{19}$$

$$U_{A_z}^{Ellipsoid}(\check{\mu}_{A_z}) = \left\{ \mu \in \mathbb{R}^{m_z} \ \left| (\mu - \check{\mu}_{A_z})^T {\check{\Sigma}_{AA_z}}^{-1} (\mu - \check{\mu}_{A_z}) \le \delta_{A_z}^2 \right.\right\}$$

$$= \left\{ \mu \in \mathbb{R}^{m_z} \left| \mu = \check{\mu}_{A_z} + \delta_{A_z} \check{\Sigma}_{AA_z}^{\frac{1}{2}} \psi_{A_z}, \left\|\psi_{A_z}\right\| \le 1 \right.\right\} \tag{20}$$

where $U_{H_z}^{Ellipsoid}$ is the confidence ellipsoid for the historical market data centred at the expected returns $\check{\mu}_{H_z}$, and $U_{A_z}^{Ellipsoid}$ is the confidence ellipsoid for the analysts' dataset centred at $\check{\mu}_{A_z}$.

The sizes of the two uncertainty sets, $\delta_{H_z} \geq 0$ and $\delta_{A_z} \geq 0$, reflect the differences in the reliability of the different data types. Usually, analysts' forecasts contain up-to-date investment information, e.g., earnings announcements news. Therefore, δ_{A_z} should be chosen to be smaller than δ_{H_z}. The investor may also choose different δ_{A_z} for different analysts $z \in \mathbb{Z}$ to reflect the quality of their inputs.

Consequently, we formulate a robust multi-analyst approach that solves:

$$\max_{x\in\mathbb{R}^n} \min_{z\in\mathbb{Z}} \min_{\mu_{H_z},\mu_{A_z}} \frac{1}{1+exp\left(-\theta_z\left(\mu_z^T x-\frac{\lambda}{2}x^T\check{\Sigma}_z x-R^{Target}\right)\right)}$$

$$subject\ to \quad \left(\mu_{H_z}-\check{\mu}_{H_z}\right)^T \check{\Sigma}_{HH_z}^{-1}\left(\mu_{H_z}-\check{\mu}_{H_z}\right) \leq \delta_{H_z}^2 \tag{21}$$

$$\left(\mu_{A_z}-\check{\mu}_{A_z}\right)^T \check{\Sigma}_{AA_z}^{-1}\left(\mu_{A_z}-\check{\mu}_{A_z}\right) \leq \delta_{A_z}^2$$

An important special case is when the investor chooses $\delta_{A_z} = 0$ for all $z \in \mathbb{Z}$. This is the scenario where the investor fully takes all advice from the financial analysts into consideration. The robust multi-analyst approach (21) reduces to

$$\left(RF_{MV}^{Seperate}\right) \quad \max_{x\in\mathbb{R}^n} \min_{z\in\mathbb{Z}} \min_{\mu_{H_z}} \frac{1}{1+exp\left(-\theta_z\left(\mu_z^T x-\frac{\lambda}{2}x^T\check{\Sigma}_z x-R^{Target}\right)\right)}$$

$$subject\ to \quad \left(\mu_{H_z}-\check{\mu}_{H_z}\right)^T \check{\Sigma}_{HH_z}^{-1}\left(\mu_{H_z}-\check{\mu}_{H_z}\right) \leq \delta_{H_z}^2 \tag{22}$$

$$\mu_{A_z} = \check{\mu}_{A_z}$$

Similar to problem (17), the robust multi-analyst approach (22) can be simplified and transformed to a simpler max-min problem below:

$$\left(RF_{MV}^{Separate}\right) \quad \max_{x\in\mathbb{R}^n} \min_{z\in\mathbb{Z}} \frac{1}{1+exp\left(-\theta_z\left(\check{\mu}_z^T x-\frac{\lambda}{2}x^T\check{\Sigma}_z x-R^{Target}-\delta_{H_z}\left\|\check{\Sigma}_{HH_z}^{\frac{1}{2}}x_H\right\|\right)\right)}$$

with $\begin{pmatrix} \check{\mu}_{H_z}-\delta_{H_z}\frac{\check{\Sigma}_{HH_z}x_H}{\left\|\check{\Sigma}_{HH_z}^{\frac{1}{2}}x_H\right\|} \\ \check{\mu}_{A_z} \end{pmatrix}$ denoting the worst-case scenario of the expected returns and $\check{\mu}_z = \begin{pmatrix} \check{\mu}_{H_z} \\ \check{\mu}_{A_z} \end{pmatrix}$.

In summary, solving problem (21) or (22) produces the selected portfolio. As shown above, this asset allocation method can address the ambiguity issue associated with analysts' recommendations by fuzzy logic. It uses the worst-case scenario analysis to deal with some over-optimistic forecasts of the analysts. In addition, by setting different confidence levels, it can appropriately handle the data heterogeneity problem such as the sampling errors of historical market data.

3. Empirical application

This section empirically assesses the performances of the developed multi-analyst method and its associated robust counterpart approach and evaluates the effects of professional analysts' recommendations on the investors' portfolio allocation decision outcomes.

3.1. Taiwanese stock market and analysts' investment recommendations

This empirical study focuses on the stocks of the top 148 listed companies representing about 90% of the total Taiwanese stock market capitalization value[3].We use historical market data and financial analysts' forecasts/recommendations for portfolio selection from April 2012 to April 2014 with 492 trading days. The Taiwanese stock market data is from DataStream Inc., whilst the financial analysts' data is collected from two top-10 Taiwanese securities brokerage firms. After pre-screening, the remaining comprises of 984 daily investment newsletters with 1,893 stock forecasts/recommendations. Figure 3 shows the first ("analyst 1") and the second ("analyst 2") securities brokerage firms, making 1,585 and 308 stock forecasts/recommendations, respectively, during the period.

[Insert Figure 3 here]

[Insert Figure 4 here]

To evaluate the performances of various investment strategies, the portfolios are constructed for every trading day with D days as the holding period. Consider, for example, the scenario where we use

[3] The market sample is based on the FTSE TWSE Taiwan 50 Index (TAISE50) and FTSE TWSE Taiwan Mid-Cap 100 Index (TAIM100). Two newly listed stocks are removed from the analysis due to insufficient historical data.

the multi-analyst method F_{MV} on day t. The following steps are taken: (a) estimation of the input parameters in (3) based on historical market data for the past 520 trading day up to day t; (b) calculation of the expected return vectors and covariance matrices in (7) and (8) and build the view models based on the analysts' investment forecasts on day t; (c) construction of the portfolio by solving problem (15). This portfolio is then held for D days and profits/losses are calculated at the end of the D-day holding period, i.e. on day $t + D$. This is repeated for each trading day t during the study period. In the following analysis, D is taken as 5 days. Fig. 4 illustrates a schematic view of the process.

3.2. *Performances of different investment strategies*

We investigate the performances of the proposed multi-analyst method F_{MV} and the robust approach $RF_{MV}^{Separate}$. For simplicity, we assume that there is no risk-free asset and short selling is prohibited for all the investment strategies considered in this section.

For comparison purposes, we also consider the following investment strategies on each trading day: (a) the portfolios constructed using the classical mean-variance portfolio selection method (1), denoted as P_{MV}; (b) the portfolios constructed by the robust counterpart approach denoted as R_{MV}; (c) the equally-weighted $(1/N)$ asset allocation (DeMiguel et al., 2009). Note that the expected return of the equally-weighted $(1/N)$ portfolio is also used as the investment benchmark R^{Target} for F_{MV} and $RF_{MV}^{Separate}$ methods.

On each trading day t, we use the above investment strategies to construct the portfolios and evaluate their performances using the ex-ante expected risk-adjusted returns, which is based on the expected portfolio returns calculated on the same day. After each D-day holding period, we carry out an out-of-sample test on day $t + D$ by calculating the ex-post realised returns of the portfolios.

The two Taiwanese securities brokerage firms from which newsletters were collected have market shares' ratio of approximately 3:1, which we then use as an indicator of their credibility. Hence, the investor chooses θ_z for each individual analyst z to be proportional to the corresponding market share: $\theta_1 = 0.7635$ for analyst 1 and $\theta_2 = 0.2365$ for analyst 2. Two types of investor are considered, investor A and investor B; each has different opinions regarding the parameter estimates from the historical market data implying different desired robustness levels for the ellipsoidal uncertainty sets. Investor A has a strong belief that historical stock performances are good signs of future performances and assigns a tighter uncertainty set for the return estimates with $\delta = \delta_{H_z} = 0.23$ for $z = 1,2$, with the true values of the stock returns to fall in the confidence ellipsoids with a 95% probability. In contrast, investor B is more hesitant to employ the historical market data to estimate the parameters and assigns $\delta = \delta_{H_z} = 1$ for $z = 1,2$ for a loose ellipsoidal uncertainty set $U^{Ellipsoid}$, where the true values of the stock returns are expected to be in the ellipsoidal uncertainty set $U^{Ellipsoid}$ with a 50% probability (Fabozzi et al., 2007).

We form the portfolios based on different risk aversion levels with the various investment strategies and report the average expected portfolio performances in Table 4. Table 4 shows the benefits of using the multi-analyst approach F_{MV} and the robust multi-analyst approach for investors A and B (denoted by $RF_{MV-A}^{Seperate}$ and $RF_{MV-B}^{Seperate}$ respectively): they achieve higher expected risk-adjusted returns than the conventional investment strategies P_{MV}, and the robust counterpart approach for investors A and B, i.e., R_{MV-A} and R_{MV-B}. The equally-weighted allocation has the lowest expected risk-adjusted returns.

[Insert Table 4 here]

Table 5 reports the realised returns calculated at the end of each 5-day holding period. From Table 5, we note that all the strategies outperform the equally-weighted portfolio $1/N$. This is not surprising: the equally-weighted portfolio is the best choice only if all assets have the same correlation coefficient as well as identical means and variances.

Table 5 also shows that the portfolio allocation decision that includes analysts' recommendation, i.e. F_{MV} , $RF_{MV-A}^{Seperate}$ and $RF_{MV-B}^{Seperate}$, leads to a better out-of-sample performance than those that do not rely on analysts' recommendations, i.e., the approaches of P_{MV}, R_{MV-A} and R_{MV-B}.

In addition, the performances also depend on whether further robustness is considered when allocating assets; investor A has a higher return than investor B. The underlying assumptions of investors A and B give rise to differences in modelling the uncertainty sets. Here investor A is assumed to have a stronger belief of the stock performance, hence leading to a tighter uncertainty set to incorporate within the portfolio allocation model. This difference in the belief of the historical market returns/performance results in different portfolio outcomes as showed by the out-of-sample, where investor A outperforms investor B. This is expected: a feature of robust optimisation is that it often leads to an overly conservative outcome (Gregory, et al, 2011).

[Insert Table 5 here]

Figure 5 shows the realised cumulative returns of these investment strategies based on risk aversion coefficient $\lambda = 0.5$ at different time periods. We can see from Figure 5 that F_{MV} constantly outperforms all the other investment strategies, whereas the equally-weighted allocation has the worst performance throughout the period. P_{MV} performs reasonably well, particularly in the second half of the time period. In addition, the robust multi-analyst portfolios $RF_{MV}^{Separate}$ outperform the conventional robust portfolios R_{MV} for both investors A and B, with investor A having a higher return than that of investor B. These observations are consistent with the findings in Table 5.

[Insert Figure 5 here]

4. Conclusions and discussion

The BL method is long regarded as a useful and practical approach to incorporating fund manager's views about the future market into investment portfolio construction. These views could also come from a third party's recommendations, such as a financial analyst consulted by retail investors. Investors may also source additional analysts' recommendations for validation purposes. This leads to a multiple information source problem.

Inspired by the BL method, we have developed a new portfolio selection model that incorporates the views of multiple analysts and adjusts for the credibility as perceived by the retail investor. We contribute to the literature by: (a) embedding the fuzzy set theory into the portfolio selection, thus addressing the ambiguity issue associated with analysts' forecasts for building the view models; (b) applying the worst-case scenario analysis to the decision-making problem, to manage several rival information sources and overcome over-optimistic analysts' forecasts; and (c) employing uncertainty sets in the robust counterpart approach to deal with the heterogeneity of data from disparate sources.

We note that the original BL method proposed by Black and Litterman (1991, 1992) is a Bayesian approach (Meucci, 2010; Schöttle et al., 2010). Relating to the Bayesian statistics literature, an early work in West and Crosse (1992) concerns investment predictions by multiple analysts, an area of increasing interest in the literature (Aastveit et al, 2018; McAlinn and West, 2018). These studies consider time series analysis with dynamic linear models and investigate Bayesian predictive synthesis for combining multiple agents' opinions to improve forecasts. These approaches are quite different from the single-period analytical approach presented in this paper, but as a future research area, could be applied by using retail investors data combining with multiple agents' opinions.

Our empirical study uses nearly 1000 daily newsletters from two top-10 Taiwanese brokerage firms, to evaluate the proposed multi-analyst approach. Comparing our results with existing methods based only on historical market data, the developed methods have better performances, hence supporting the use of analysts' recommendations in the portfolio construction.

Our study also provides an improvement on the outcomes of robust optimisation by including multiple analysts' views in the portfolio allocation. It is known that robust optimisation often leads to an overly conservative outcome (see, e.g. Gregory et al, 2011), but with multiple analysts' views, the uncertainty from the sampling errors is reduced; this in turn improves the performance of robust optimisation.

In this paper, the possibility of herding behaviour among multiple analysts is not addressed. Trueman (1994) describes this as a case where analysts release forecasts similar to those previously announced by other analysts, even when this is not justified by their information. Recent studies show that releasing optimistic information in times of high market sentiment reduces herding practices, whereas herding increases in difficult situations when analysts have to release negative information (Blasco et al., 2018). The herding effect could be addressed in future research by constructing a dynamic model for the joint prior distribution of the analysts' views that will account for the potential correlations among different analysts.

This paper supports the role of retail investors in portfolio construction using analysts' recommendations. Recently, similar topics on asset selection of retail investors have attracted increasing attention. For example, Kelley and Tetlock (2013) show that collectively retail investors can predict monthly returns, and hence are not entirely 'noise' traders, as assumed in the finance literature. Using online recommendations, retail investors can source information more efficiently than in the past, thus reinforcing their ability to select investment portfolios. This paper has demonstrated that using analysts' recommendations can enhance the realistic return of the investment.

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Appendix. Propositions and proofs

Proposition 1. Consider a single-period portfolio selection problem with n risky assets and Z analysts, each providing the investment forecasts characterised by equations (10)-(12). Then for an investor who chooses his/her portfolio by solving problem (15), the optimal portfolio x^* is given by

$$x^* = \frac{1}{\lambda} {\Sigma^*}^{-1} \mu^*$$

with

$$\Sigma^* = \sum_{z=1}^{Z} \omega_z \check{\Sigma}_z \qquad \text{and} \qquad \mu^* = \sum_{z=1}^{Z} \omega_z \check{\mu}_z ,$$

where $\omega_z = \theta_z\phi_z / \sum_{z=1}^{Z} \theta_z\phi_z$, and $\phi = [\phi_1, \dots, \phi_Z]^T \in \mathbb{R}^Z$ is the vector of the Lagrange multipliers.

Proof: The portfolio allocation problem $(\boldsymbol{F_{MV}})$ is equivalent to

$$\begin{aligned} &\max_{x\in\mathbb{R}^n,\ \zeta\in\mathbb{R}} && \zeta \\ &\text{subject to} && \zeta \leq \frac{1}{1+\exp\left(-\theta_z\left(\breve{\mu}_z{}^T x - \frac{\lambda}{2}x^T\breve{\Sigma}_z x - R^{Target}\right)\right)} \quad \text{for all } z \in \mathbb{Z}. \end{aligned} \tag{A1}$$

Let $\eta = \log(\zeta/(1-\zeta))$. It follows immediately that solving problem (A1) is equivalent to

$$\begin{aligned} &\max_{x\in\mathbb{R}^n,\ \zeta\in\mathbb{R}} && \eta \\ &\text{subject to} && \eta \leq \theta_z\left(\breve{\mu}_z{}^T x - \frac{\lambda}{2}x^T\breve{\Sigma}_z x - R^{Target}\right) \quad \text{for all } z \in \mathbb{Z}. \end{aligned} \tag{A2}$$

Now, let $g_z(x) = \theta_z\left(\breve{\mu}_z{}^T x - \frac{\lambda}{2}x^T\breve{\Sigma}_z x - R^{Target}\right)$ for each $z \in \mathbb{Z}$. We form the Lagrangian function to problem (A2) as $\mathcal{L}(\eta, x, \phi) = \eta - \sum_{z=1}^{Z}\phi_z(\eta - g_z(x))$, where $\phi = [\phi_1, \dots, \phi_Z]^T \in \mathbb{R}^Z$ is a vector of the Lagrange multipliers. We can verify that the partial derivatives of the Lagrangian function with respect to variables η and x are $\frac{\partial\mathcal{L}}{\partial\eta} = 1 - \sum_{z=1}^{Z}\phi_z$ and $\frac{\partial\mathcal{L}}{\partial x} = \sum_{z=1}^{Z}\phi_z g_z{}'(x)$, where $g_z{}'(x) = \theta_z\breve{\mu}_z - \lambda\theta_z\breve{\Sigma}_z x$. Hence, according to the Karush–Kuhn–Tucker conditions, the corresponding conditions for the optimal portfolio x^* of problem (F_{MV}^*) are

$$1 - \sum_{z=1}^{Z}\phi_z = 0, \qquad \text{and} \qquad \sum_{z=1}^{Z}\phi_z g_z{}'(x^*) = 0,$$

$$\phi_z(\eta - g_z(x^*)) = 0, \qquad \text{and} \qquad \phi_z \geq 0.$$

It can be verified that the Lagrange multipliers $\phi \in \mathbb{R}^Z$ must satisfy $0 \leq \phi_z \leq 1$ for all $z \in \mathbb{Z}$. In addition, we can rearrange equation $\phi_z(\eta - g_z(x^*)) = 0$ to obtain $\sum_{z=1}^{Z}\phi_z\theta_z\breve{\mu}_z = \lambda\sum_{z=1}^{Z}\phi_z\theta_z\breve{\Sigma}_z x^*$. Consequently, the optimal portfolio x^* of the portfolio selection problem is given by $x^* = \frac{1}{\lambda}\,\Sigma^{*-1}\mu^*$ with $\Sigma^* = \sum_{z=1}^{Z}\theta_z\phi_z\breve{\Sigma}_z$ and $\mu^* = \sum_{z=1}^{Z}\theta_z\phi_z\breve{\mu}_z$. Finally, we normalise the weight ω_z by $\omega_z = \theta_z\phi_z/\sum_{z=1}^{Z}\theta_z\phi_z > 0$ so that $\sum_{z=1}^{Z}\omega_z = 1$. Clearly, the normalisation does not affect the solution $x^* = \frac{1}{\lambda}\,\Sigma^{*-1}\mu^*$. □

Proposition 2. The robust counterpart approach (17) is equivalent to the following max-min problem

$$\left(RF_{MV}^{Joint}\right) \quad \max_{x\in\mathbb{R}^n} \min_{z\in\mathbb{Z}} \frac{1}{1+exp\left(-\theta_z\left(\breve{\mu}_z^T x-\frac{\lambda}{2}x^T\breve{\Sigma}_z x-R^{Target}-\delta_0\left\|\breve{\Sigma}_z^{\frac{1}{2}}x\right\|\right)\right)}$$

with $\breve{\mu}_z - \delta_0 \frac{\Sigma_z x}{\left\|\Sigma_z^{1/2} x\right\|}$ denoting the worst-case scenario of the expected returns.

Proof: For the uncertainty set of the expected returns based on the forecasts of analyst $\boldsymbol{z} \in \mathbb{Z}$, $\boldsymbol{U_z^{Ellipsoid}}(\boldsymbol{\breve{\mu}_z}) = \left\{\boldsymbol{\mu_z} \in \mathbb{R}^{\boldsymbol{n}} \middle| \boldsymbol{\mu_z} = \boldsymbol{\breve{\mu}_z} + \boldsymbol{\delta_0}\boldsymbol{\breve{\Sigma}_z^{\frac{1}{2}}\psi}, \|\boldsymbol{\psi}\| \le \boldsymbol{1}\right\}$, the problem (17) can be rearranged as

$$\max_{x\in\mathbb{R}^n} \min_{z\in\mathbb{Z}} \min_{\|\psi\|\le 1} \frac{1}{1+exp\left(-\theta_z\left(\left(\breve{\mu}_z+\delta_0\breve{\Sigma}_z^{\frac{1}{2}}\psi\right)^T x-\frac{\lambda}{2}x^T\breve{\Sigma}_z x-R^{Target}\right)\right)}$$

$$= \max_{x\in\mathbb{R}^n} \min_{z\in\mathbb{Z}} \frac{1}{1+exp\left(-\theta_z\left(\breve{\mu}_z^T x-\frac{\lambda}{2}x^T\breve{\Sigma}_z x-R^{Target}+\delta_0\min_{\|\psi\|\le 1}\psi^T\hat{\Sigma}^{\frac{1}{2}}x\right)\right)}$$

Since $\psi^T\hat{\Sigma}^{\frac{1}{2}}x$ is minimised at $\psi^* = -\frac{\Sigma_z^{1/2}x}{\left\|\Sigma_z^{1/2}x\right\|}$, the above max-min problem becomes to (18). □

Table 1.

<table>
<tr><td colspan="2">XXXX TECHNOLOGY</td><td>In December last year, the company reported a 48.46% year-by-year increase in monthly revenue to NT$ 94.1 million.</td></tr>
<tr><td>Closing Price:</td><td>93.3</td><td rowspan="3">The company is now reaping the harvest of the touch panel products in the mainland China market, and the market share has been gradually increased. Therefore, the annual revenue is expected to reach another new height this year.</td></tr>
<tr><td>Price Resistance:</td><td>100</td></tr>
<tr><td>Price Support:</td><td>89</td></tr>
</table>

This table provides an example of a forecast on a stock, XXXX technology using the resistance and support approach to interpret analyst's recommendation, where the interpretation based on the fuzzy variable input is a quadruplet form $(1.285\%, 7.18\%, -5.895\%, 0\%)$. This is based on the discussion of case (1) in section 2.2.2.

Table 2.

XXXX BANK		Neutral	Analyst A
Closing Price	17.5	Remain "Neutral" rating for XXXX BANK with a NT$ 17.8 price target. We retain the recommended investment strategy for the stock as "Neutral" based on the following considerations. First, XXXX BANK is one of the largest domestic banks in Taiwan in terms of enterprise size, and it has a relatively decent market share in the Taiwanese financial service sector. Although…"	
Target Price	17.8		
Potential %	2%		

This table provides an example of a forecast on stock, XXXX Bank, using the target price and the potential rate approach to interpret analyst's recommendation, where the fuzzy interpretation is $(1.71\%, 1.71\%, 1.71\%, 0.29\%)$. This is based on the discussion of case (2) in section 2.2.2.

Table 3

<table>
<tr><th>Company</th><th>1st Price Boundary</th><th>2nd Price Boundary</th><th>Closing Price</th><th>3rd Price Boundary</th><th>4th Price Boundary</th></tr>
<tr><td>XXXX CONSTRUCTION</td><td>32.2</td><td>33.6</td><td>34.3</td><td>35.5</td><td>36.35</td></tr>
<tr><td colspan="6">The share price is falling due to its disappointing quarterly revenue and the lag effect after the ex-dividend date.</td></tr>
<tr><td>XXXX INTERNATIONAL</td><td>7.0</td><td>7.65</td><td>7.8</td><td>7.9</td><td>8.08</td></tr>
<tr><td colspan="6">With the reported quarterly net losses, it is likely that the share price may drop below the last trend line bottom.</td></tr>
</table>

This table provides two examples of stock forecasts. Both use the price boundaries approach to interpret analyst's recommendations. The fuzzy interpretations are $(-2.04\%, 3.50\%, 4.08\%, 2.48\%)$ for XXXX Construction and $(-1.92\%, 1.28\%, 8.33\%, 2.31\%)$ for XXXX International. This approach is based on the discussion of case (3) in section 2.2.2.

Table 4.

	With analysts' recommendations			Without analysts' recommendations			
Risk Aversion	F_{MV}	$RF_{MV-A}^{Seperate}$	$RF_{MV-B}^{Seperate}$	P_{MV}	R_{MV-A}	R_{MV-B}	$1/N$
$\lambda = 0$	0.4034	0.4924	0.5665	0.3566	0.4253	0.4126	
$\lambda = 0.5$	0.6420	0.6304	0.5476	0.5597	0.5134	0.3303	
$\lambda = 1$	0.5597	0.5560	0.4465	0.4781	0.4284	0.2828	0.0631
$\lambda = 3$	0.9602	0.9426	0.5778	0.3063	0.2816	0.2038	
$\lambda = 5$	0.9450	0.9041	0.5566	0.2595	0.2408	0.1773	

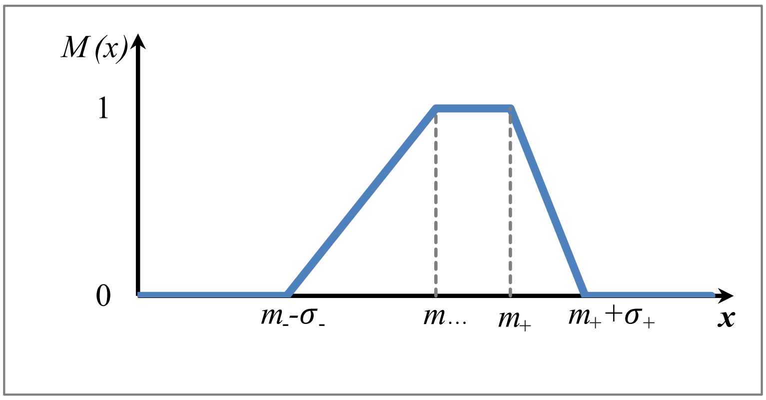
This table shows the ex-ante expected risk-adjusted returns (Mean/SD) of the portfolios selected using different investment strategies, F_{MV} , $RF_{MV-A}^{Seperate}$, $RF_{MV-B}^{Seperate}$, P_{MV} , R_{MV-A} , R_{MV-B} and 1/N. They are compared across different risk aversion levels, with and without analysts' recommendations, except for 1/N.

Table 5.

	With analysts' recommendations			Without analysts' recommendations			
Risk Aversion	F_{MV}	$RF_{MV-A}^{Seperate}$	$RF_{MV-B}^{Seperate}$	P_{MV}	R_{MV-A}	R_{MV-B}	$1/N$
$\lambda = 0$	1.2803	**1.2845**	0.5742	1.1770	1.2394	0.4825	
$\lambda = 0.5$	**0.7089**	0.6104	0.4520	0.6535	0.5590	0.3893	
$\lambda = 1$	**0.4961**	0.4648	0.3994	0.4623	0.4258	0.3486	0.2382
$\lambda = 3$	**0.3508**	0.3506	0.3488	0.3180	0.3138	0.3028	
$\lambda = 5$	0.3412	**0.3440**	0.3419	0.2957	0.2940	0.2891	

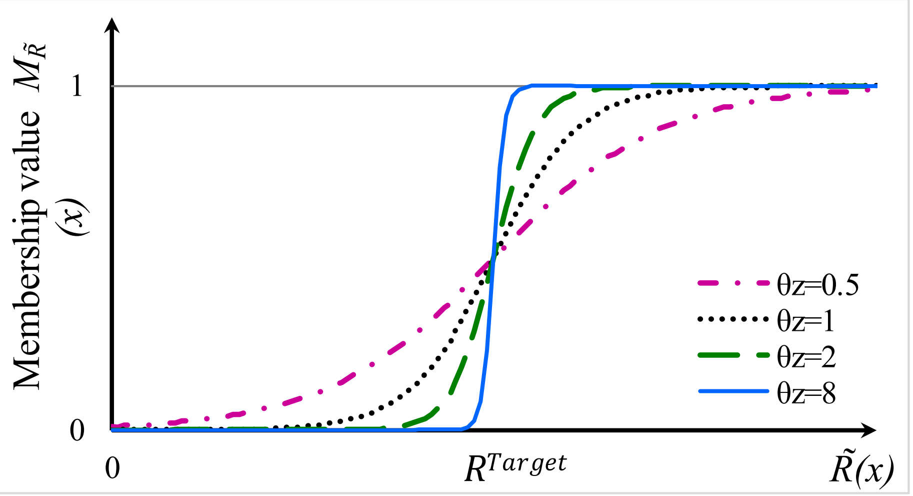
This table shows the ex-post realised returns of the portfolios selected using different investment strategies, F_{MV} , $RF_{MV-A}^{Seperate}$, $RF_{MV-B}^{Seperate}$, P_{MV} , R_{MV-A} , R_{MV-B} and 1/N. They are compared across different risk aversion levels, with and without analysts' recommendations, except for 1/N. The numbers reported in bold shows the highest return in each of the risk aversion level for the all strategy types.

Figure 1



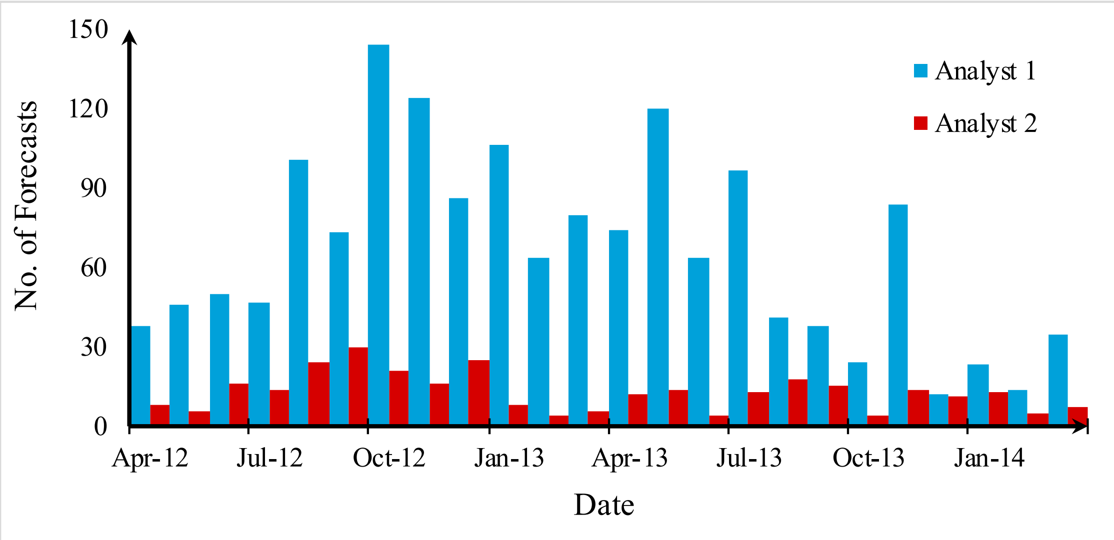
This figure graphically presents the membership function based on equation (6). The membership function is a model used to represent/interpret the analyst's investment forecasts which is regarded as qualitative in this paper.

Figure 2



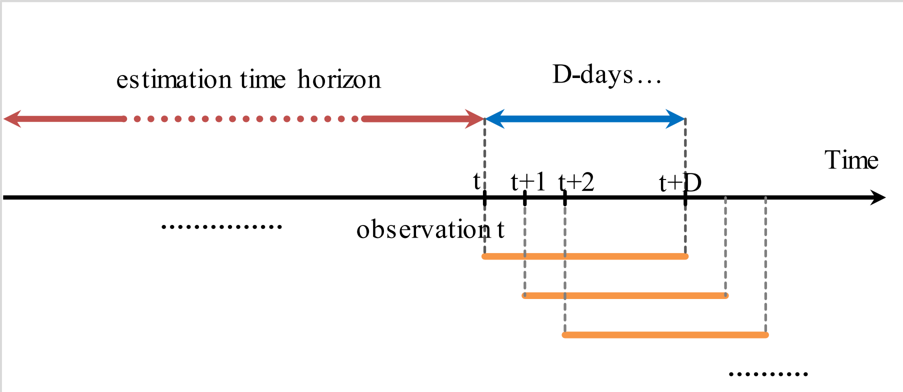
This figure graphically shows the difference in the various membership functions arising from the different credibility levels of the analysts as perceived by the investor. The membership functions are based on the target returns as the benchmark of the investor and used as the aspiration level for the portfolio.

Figure 3



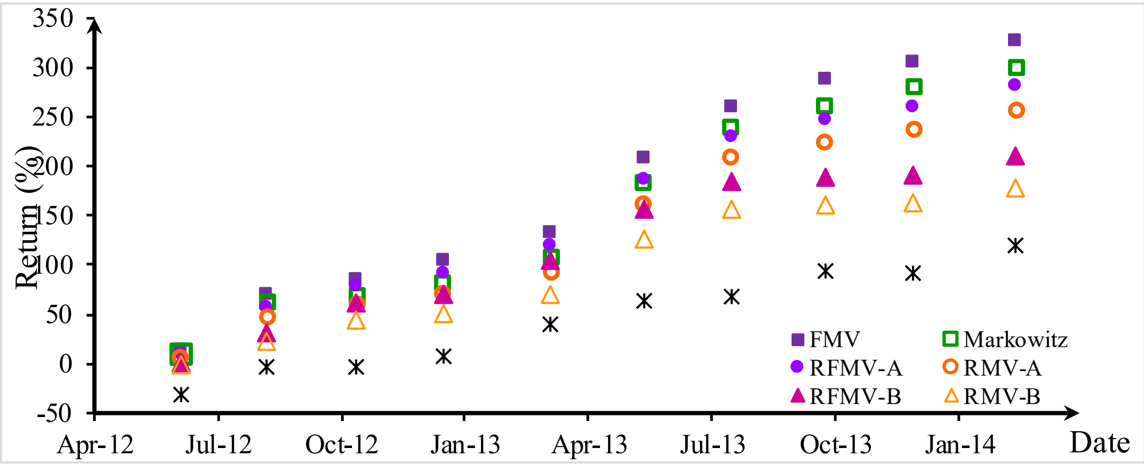
This figure shows the monthly volumes of stock recommendations provided by two securities brokerage firms, analysts 1 and 2. They provided a total of 1585 (analyst 1) and 308 (analyst 2) stock recommendations over the two years study period April 2012 to April 2014.

Figure 4



This figure shows the process of portfolio selection. At each time point t, 520 trading days up to time t are used to estimate the input parameters based on equation (3). Then, expected return and variances are estimated based on equations (7) and (8) to build the "view" model using the analysts' forecasts on day t. This is followed by the construction of the full portfolio. The portfolio is then held for D (i.e., 5) days and profit/loss are calculated at the end of D day. This is repeated by moving one-day ahead each time from *t* keeping estimation periods of 520 days each time.

Figure 5



This figure shows the realised cumulative returns of portfolios based on various investment strategies, i.e., F_{MV}, $RF_{MV-A}^{Seperate}$, $RF_{MV-B}^{Seperate}$, P_{MV}, R_{MV-A} and R_{MV-B}.