The responses of long moving vertical ropes and cables subjected to dynamic loading due to the host structure sway

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Abstract. Long moving vertical ropes and cables are used in tall slender engineering structures. For example, in high-rise buildings and towers they are employed as a means of car and counterweight suspension and for compensation of tensile forces over the traction sheave in traction drive elevators. An adverse situation arises when the host structure is excited near its fundamental natural frequency and sways harmonically at low frequencies. This often results in a passage through resonance conditions in the rope system when the slowly varying natural frequencies of the ropes approach the frequency of the inertial load resulting from the structure sway. The nature of such a loading is usually nondeterministic and it is necessary to apply stochastic models to analyze the dynamic responses of the ropes. In this paper a model to describe the lateral dynamic behaviour of a vertical moving rope hosted in a tall slender structure is developed. The model takes into account the fact that the longitudinal elastic stretching of the ropes is coupled with their transverse motions which results in cubic nonlinear terms. The mathematical model comprises nonstationary nonlinear ordinary differential equations. Taking into account the fact that the motion of the structure can be represented as a narrow-band process mean-square equivalent to the harmonic process, equations for the second- and higher-order joint statistical moments are obtained with the aid of Itô differential rule. Due to the non-linearities the moment equations form a system of an infinite hierarchy which needs to be truncated at some level. The technique of cumulant - neglect closure (CNC) is proposed in order to implement the truncation procedure and to treat the system numerically.

Introduction

Environmental phenomena such as strong wind conditions and earthquakes cause tall civil structures such as towers and high-rise buildings to vibrate (sway) at low frequencies and large amplitudes [1]. When the structure is sufficiently flexible the dynamic response to forces generated by these phenomena is significant. As a result, the corresponding inertial loads often excite cables and ropes that are part of equipment hosted within the structure. For example, large resonance motions of ropes and cables in high-speed elevators in high-rise buildings take place [2].

In order to investigate the passage through resonance in the rope system the excitation mechanism can be represented by a *deterministic* function and consequently the response of the system is treated as a deterministic phenomenon. However, the nature of loading caused by wind is usually *nondeterministic* (stochastic). The excitation should then be described by a *stochastic process* so that the methods of stochastic dynamics could be employed to predict the dynamic behaviour of the rope system [3].

In this paper, the results of a study to predict the dynamic response of long ropes moving at speed within a tall slender host structure are presented. First the deterministic model is developed to describe the dynamic behavior of the system. Then, taking into account the stochastic nature of the building motion the nonstationary nonlinear differential equations governing the statistical moments of the state vector are developed. Due to the cubic nonlinearities these equations form an infinite hierarchy and the closure technique is proposed so that they could be solved numerically.

Dynamic Model and Equations of Motion

The model depicted in Fig. 1 is used to study the dynamic response of a vertical rope of time-varying length $L(t)$. The top end of the rope which has mass per unit length m, Young's modulus E and an effective cross-sectional area A_{eff} is attached at B to a support moving at speed v within a tall cantilevered host structure. The host structure sways resulting in motion $w_0(t)$ of amplitude A_0 at the level defined by the coordinate z_0 measured from the base level. The deformations of the structure are described as $A_0\Psi(z)$ where $\Psi(z)$ represents the shape function with z denoting a coordinate measured from the base level. The sway results in an inertial dynamic load acting upon the rope and its corresponding dynamic response is represented by the lateral displacements denoted as $w(x,t)$, where x is measured from the origin O placed at distance l below the base level. The lateral response $w(x,t)$ is coupled with axial (longitudinal) motions of the rope that are denoted as $u(x,t)$.

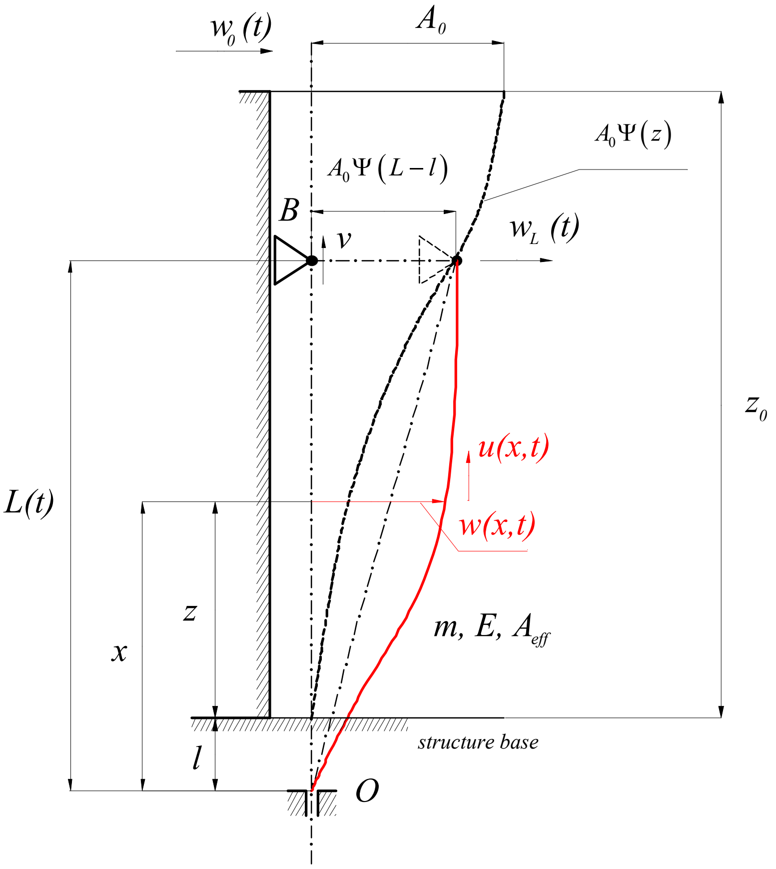

Fig. 1 Vertical rope system

The mean quasi-static tension of the rope is expressed as

$$T^i(x) = T_0 + m(g+a) \tag{1}$$

where T_0 represents a constant tension term, $a=\dot{v}$ is the acceleration of the support (an overdot denotes the time derivative) and g is the acceleration of gravity. The equations governing the undamped dynamic displacements $u(x,t)$ and $w(x,t)$ of the rope can be developed by applying Hamilton's principle which requires that

$$\int_{t_1}^{t_2}\left(\delta E_k - \delta \Pi_e - \delta \Pi_g\right)dt = 0 \tag{2}$$

where E_k, Π_e and Π_g denote the kinetic energy, the potential elastic energy and the potential gravitational energy.

The kinetic energy of the rope is given as

$$E_k = \frac{1}{2}\int_0^L m\left(u_t^2 + w_t^2\right)dx \tag{3}$$

where $(\,)_t$ denotes partial derivative with respect to time. The elastic potential energy of the rope is expressed as

$$\Pi_e = \Pi_e^i + \int_0^L \left(T^i + \frac{1}{2}EA\varepsilon\right)\varepsilon dx \tag{4}$$

where Π_e^i is the initial (constant) strain energy and ε represents the strain measure due to stretching of the rope and is given as

$$\varepsilon = u_x + \frac{1}{2}w_x^2 \tag{5}$$

where $(\,)_x$ denotes partial derivative with respect to x. The gravitational potential energy of the rope is expressed as

$$\Pi_g = mg\int_0^L u\left(x,t\right)dx \tag{6}$$

Using equations (3) – (6) in the Hamilton's formulation (2) yields the equations of motion describing the longitudinal and lateral displacements over the spatial domain $0 < x < L\left(t\right)$ as follows

$$m\frac{d^2u}{dt^2} - EA\left(u_x + \frac{1}{2}w_x^2\right)_x - ma = 0 \tag{7}$$

$$m\frac{d^2w}{dt^2} - \left[T_0 + m\left(g+a\right)x\right]w_{xx} - m\left(g+a\right)w_x - EA\left[\left(u_x + \frac{1}{2}w_x^2\right)w_x\right]_x = 0 \tag{8}$$

where

$$\frac{d^2u}{dt^2} = u_{tt} + 2vu_{xt} + v^2u_{xx} + au_x \tag{9}$$

$$\frac{d^2w}{dt^2} = w_{tt} + 2vw_{xt} + v^2w_{xx} + aw_x \tag{10}$$

The displacements and the boundaries $x = 0, L$ are defined as

$$u(0,t)=u(L,t)=0 \tag{11}$$

$$w(0,t)=0,\ \ w(L,t)=w_L\left(t\right) \tag{12}$$

In this study only the lower-order lateral modes are considered. Thus, bearing in mind that no interaction will take place between these lateral modes and the axial (longitudinal) modes, the axial inertia terms given by equation (9) can be neglected so that equation (7) is re-written as

$$EA\left(u_x+\frac{1}{2}w_x^2\right)_x+ma=0 \tag{13}$$

which yields

$$u_x+\frac{1}{2}w_x^2=-\frac{ma}{EA}x+e(t) \tag{14}$$

Integrating (14) and using the boundary conditions (11) gives

$$e(t)=\frac{maL}{2EA}+\frac{1}{2L}\int_0^L w_x^2dx \tag{15}$$

Combining equations (14), (15) and using them together with equation (10) in the equation of lateral motions (8) yields

$$mw_{tt}-\left[T_0+m\left(\frac{1}{2}aL-v^2+gx\right)+\frac{EA}{2L}\int_0^L w_x^2dx\right]w_{xx}+m\left(a-g\right)w_x+2mvw_{xt}=0 \tag{16}$$

In order to accommodate the excitation due to the structure sway in the equation of motion (16) the overall lateral displacement of the rope is expressed as

$$w(x,t)=\bar{w}(x,t)+\frac{x}{L\left(t\right)}w_L\left(t\right),\ \ 0\le x\le L\left(t\right) \tag{17}$$

Noting that the lateral displacement at x = L can be expressed as $w_L(t)=\Psi_L w_0\left(t\right)$, where $\Psi_L=\Psi\left(L-l\right)$, the transformation (17) is given as

$$w(x,t)=\bar{w}(x,t)+\Psi_L\frac{x}{L\left(t\right)}w_0\left(t\right),\ \ 0\le x\le L\left(t\right) \tag{18}$$

In this analysis the deformation shape function $\Psi\left(z\right)$ is assumed to be related to the fundamental mode of the structure and is approximated by a cubic polynomial as follows

$$\Psi\left(z\right)=3\left(\frac{z}{z_0}\right)^2-2\left(\frac{z}{z_0}\right)^3 \tag{19}$$

so that

$$\Psi_L=3\left(\frac{L\left(t\right)-l}{z_0}\right)^2-2\left(\frac{L\left(t\right)-l}{z_0}\right)^3 \tag{20}$$

Using transformation (18) in equation (16) results in the following

$$m\bar{w}_{tt}-\left[T+m\left(gx-v^2\right)+\frac{EA}{L}\left(\frac{1}{2}\int_0^L \bar{w}_x^2 dx+\frac{\Psi_L}{L}\int_0^L \bar{w}_x dx+\frac{\Psi_L^2}{2L}w_0^2\left(t\right)\right)\right]\bar{w}_{xx} \\ +m\left(a-g\right)\bar{w}_x+2mv\bar{w}_{xt}=f_{\bar{w}}\left(x,t\right) \tag{21}$$

where $T\left(L\left(t\right)\right)=T_0+\frac{1}{2}maL\left(t\right)$ and $f_{\bar{w}}\left(x,t\right)$ represents excitation due to the structure sway given as

$$f_{\bar{w}}\left(x,t\right)=m\frac{\Psi_L}{L\left(t\right)}\left[\left(g-a\right)w_0\left(t\right)-2v\dot{w}_0\left(t\right)-\ddot{w}_0\left(t\right)\right] \tag{22}$$

The dynamic response of the rope is then approximated by the following expansion

$$\bar{w}(x,t)=\sum_{n=1}^{N}\Phi_n(x)q_n(t) \tag{23}$$

where $\Phi_n(x;L\left(t\right))=\sin\frac{n\pi}{L\left(t\right)}x,\ \ n=1,2,\ldots,N$, are the natural vibration modes of the corresponding taut string of length $L = L(t)$ with tension $T(L(t))$, and $q_n(t)$ represent the modal coordinates. By substituting series (23) into the equation of motion (21), integrating over the domain $0<x<L\left(t\right)$ with the mode shape functions $\Phi_n\left(x;L\left(t\right)\right)$ acting as the *weighting functions*, after orthogonalising with respect to the natural modes, the following system of ordinary differential equations describing the dynamic behaviour of the rope is obtained

$$\ddot{q}_r+2\zeta_r\omega_r\dot{q}_r+\lambda_r^2\left[\bar{c}^2-v^2+\frac{1}{2}c^2\Psi_L^2\left(\frac{w_0}{L}\right)^2\right]q_r+\sum_{n=1}^{N}K_{rn}q_n+\sum_{n=1}^{N}C_{rn}\dot{q}_n+\left(\frac{\lambda_r}{2}c\right)^2 q_r\sum_{n=1}^{N}\lambda_n^2q_n^2=Q_r\left(t\right) \tag{24}$$

where modal damping represented by the damping ratios ζ_r ($r=1,2,\ldots,N$) has been added, ω_r are the undamped natural frequencies of the rope, $\bar{c}=\sqrt{\frac{T\left(L\left(t\right)\right)}{m}}$ and $c=\sqrt{\frac{EA}{m}}$ represent the lateral wave speed and the longitudinal wave speed, respectively, $\lambda_r=\frac{r\pi}{L\left(t\right)}$ and the coefficients K_{rn}, C_{rn} and the modal excitation function Q_r are given as

$$K_{rn}\left(t\right)=\frac{2g}{L\left(t\right)}\begin{cases}\dfrac{r^2\pi^2}{4}, & n=r\\ \left[\left(\dfrac{a}{g}-1\right)\dfrac{nr}{n^2-r^2}+\dfrac{2rn^3}{\left(n^2-r^2\right)^2}\right]\left[\left(-1\right)^{r+n}-1\right], & n\neq r\end{cases}$$

$$C_{rn}\left(t\right)=\frac{4v}{L\left(t\right)}\begin{cases}0, & n=r\\ \dfrac{nr}{n^2-r^2}\left[\left(-1\right)^{r+n}-1\right], & n\neq r\end{cases} \tag{25}$$

$$Q_r(t)=\frac{2\Psi_L}{r\pi L\left(t\right)}\left\{\left(-1\right)^r L\left(t\right)\ddot{w}_0\left(t\right)-\left[\left(g-a\right)w_0\left(t\right)-2v\dot{w}_0\left(t\right)\right]\left[\left(-1\right)^r-1\right]\right\}$$

where $r, n = 1, 2, \ldots, N$. Equations (24) have time-varying coefficients and include cubic geometric nonlinearities arising due to the effect of rope stretching. It is evident that the host structure motion results in modal external excitation terms and also appears in the stiffness terms as *parametric excitation.*

An adverse situation arises when the structure is excited near its fundamental natural frequency. This in turn leads to a passage through resonance in the rope system when one of its time-varying natural frequencies approaches that of the inertial load resulting from the resonance sway.

In this study the dynamic deformation $w_0(t)$ is assumed to be related to the fundamental mode of the structure of frequency Ω_0. The behavior of the system within the resonance region can then be investigated using an approximated single-mode model obtained from equation (24) as follows

$$\ddot{q}_r + 2\zeta_r\omega_r\dot{q}_r + \lambda_r^2\left[\bar{c}^2 - v^2 + \frac{1}{2}gL(t) + \frac{1}{2}c^2\Psi_L^2\left(\frac{w_0(t)}{L(t)}\right)^2\right]q_r + \left(\frac{c}{2}\right)^2\lambda_r^4 q_r^3 = Q_r(t) \tag{26}$$

where q_r denotes the generalized coordinate corresponding to the resonance mode. Equation (26) is used in the stochastic analysis to follow.

Narrow-Band Stochastic Model

Motion $w_0(t)$ is assumed to have a narrow banded time variation with the centre frequency Ω_0. Thus, $w_0(t)$ is a narrow-band process mean-square equivalent to the harmonic process with the amplitude A_0 and the frequency Ω_0. It should be noted that the function $w_0(t)$ together with its first and second derivatives $\dot{w}_0(t)$ and $\ddot{w}_0(t)$ must be continuous [4]. This scenario is adequately idealized by assuming that the motion $w_0(t)$ is the response of the second order auxiliary filter to the process *X(t)*, which in turn is the response of the first-order filter to the Gaussian white noise excitation $\xi(t)$ as illustrated in Figure 2. The governing equations are

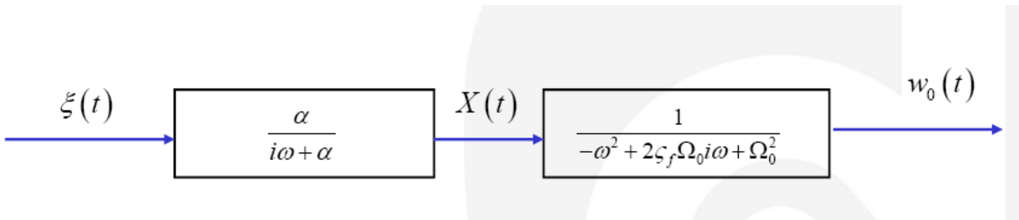

Figure 2. Narrow-band process scenario.

$$\begin{aligned}&\ddot{w}_0(t) + 2\zeta_f\Omega_0\dot{w}_0(t) + \Omega_0^2 w_0(t) = X(t)\\&\dot{X}(t) + \alpha X(t) = \alpha\sqrt{2\pi S_0}\,\xi(t)\end{aligned} \tag{27}$$

where ζ_f denotes the damping ratio of the filter which defines its band width, α is the filter variable, S_0 is the constant level of the power spectrum of white noise $\xi(t)$. Precisely speaking, the response *X(t)* to the Gaussian white noise excitation $\xi(t)$ is a process which is not differentiable in the usual sense, hence the notation $\dot{X}(t)$ is not mathematically meaningful. The second equation in (27) has a clear mathematical sense when it is written in terms of the differential. Accordingly, the governing equations for both filters are written down in a state space form as

$$d\begin{bmatrix} X \\ w_0 \\ \dot{w}_0 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -\Omega_0^2 & -2\zeta_f\Omega_0 \end{bmatrix}\begin{bmatrix} X \\ w_0 \\ \dot{w}_0 \end{bmatrix}dt + \begin{bmatrix} \alpha\sqrt{2\pi S_0} \\ 0 \\ 0 \end{bmatrix}dW(t) \tag{28}$$

where $dW(t)$ is the increment of the Wiener process $W(t)$ and the Gaussian white noise excitation $\xi(t)$ is the generalized (not in usual sense) derivative of the Wiener process.

It is evident from the last expression in equation (25) that the excitation function $Q_r(t)$ can be expressed in terms of the motion $w_0(t)$ and its time derivatives as

$$Q_r(t)=\beta_r^{(1)}w_0(t)+\beta_r^{(2)}\dot{w}_0(t)+\beta_r^{(3)}\ddot{w}_0(t) \tag{29}$$

Using equation (27) the second derivative of $w_0(t)$ can be expressed as

$$\ddot{w}_0(t)=X(t)-\Omega_0^2 w_0(t)-2\zeta_f\Omega_0\dot{w}_0(t) \tag{30}$$

so that equation (29) can be rewritten as

$$Q_r(t)=\gamma_r^{(1)}w_0(t)+\gamma_r^{(2)}\dot{w}_0(t)+\beta_r^{(3)}X(t) \tag{31}$$

where

$$\begin{aligned}\gamma_r^{(1)}&=\beta_r^{(1)}-\Omega_0^2\beta_r^{(3)}\\ \gamma_r^{(2)}&=\beta_r^{(2)}-2\zeta_f\Omega_0\beta_r^{(3)}\end{aligned} \tag{32}$$

The state vector defined as

$$\mathbf{Y}(t)=\begin{bmatrix}q_r(t) & \dot{q}_r(t) & X(t) & w_0(t) & \dot{w}_0(t)\end{bmatrix}^T \tag{33}$$

is then governed by the following set of stochastic equations

$$d\mathbf{Y}(t)=\mathbf{c}(\mathbf{Y}(t))dt+\mathbf{b}dW(t) \tag{34}$$

where

$$\mathbf{c}(\mathbf{Y})=\begin{bmatrix}\dot{q}_r\\ -K_r q_r-2\zeta_r\omega_r\dot{q}_r-\left(\frac{c}{2}\right)^2\lambda_r^4 q_r^3+\gamma_r^{(1)}w_0+\gamma_r^{(2)}\dot{w}_0+\beta_r^{(3)}X\\ -\alpha X\\ \dot{w}_0\\ X-\Omega_0^2 w_0-2\zeta_f\Omega_0\dot{w}_0\end{bmatrix} \tag{35}$$

$$\mathbf{b}=\begin{bmatrix}0 & 0 & \alpha\sqrt{2\pi S_0} & 0 & 0\end{bmatrix}^T \tag{36}$$

represent the *drift vector* and the *diffusion vector*, respectively, where

$$K_r=\lambda_r^2\left[\bar{c}^2-v^2+\frac{1}{2}gL(t)+\frac{1}{2}c^2\Psi_L^2\left(\frac{w_0(t)}{L(t)}\right)^2\right] \tag{37}$$

The elements of the drift vector (35) include polynomial non-linearity of a cubic form. It can be noted that if the non-linear term is neglected equations (34) assume the following linear form

$$d\mathbf{Y}(t)=\mathbf{A}\mathbf{Y}(t)dt+\mathbf{b}dW(t) \tag{38}$$

where

$$\mathbf{A}=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -K_r & -2\zeta_r\omega_r & \beta_r^{(3)} & \gamma_r^{(1)} & \gamma_r^{(2)} \\ 0 & 0 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\Omega_0^2 & -2\zeta_f\Omega_0 \end{bmatrix} \tag{39}$$

In the general case of polynomial non-linearity of a cubic form it is convenient to express the elements of the drift vector (35) as

$$c_i\left(\mathbf{Y}\right)=e_{im}Y_m+f_{imn}Y_mY_n+g_{imnp}Y_mY_nY_p \tag{40}$$

Equations for the mean values $\mu_i\left(t\right)=E\left[Y_i\left(t\right)\right]$, $i=1,2,\ldots,5$, of the state variables $Y_i\left(t\right)$ are then given as

$$\frac{d}{dt}\mu_i\left(t\right)=E\left[c_i\left(\mathbf{Y}\left(t\right)\right)\right],\ \ i=1,2,\ldots,5 \tag{41}$$

With the mean values expressed by (41) the following equations for the centralized (zero-mean) state variables given as $Y_i^0\left(t\right)=Y_i\left(t\right)-\mu_i\left(t\right),\ \ i=1,2,\ldots,5$, result

$$d\mathbf{Y}^0\left(t\right)=\mathbf{c}^0\left(\mathbf{Y}\left(t\right)\right)dt+\mathbf{b}dW\left(t\right) \tag{42}$$

where

$$\mathbf{c}^0\left(\mathbf{Y}\left(t\right),t\right)=\mathbf{c}\left(\mathbf{Y}\left(t\right)\right)-E\left[\mathbf{c}\left(\mathbf{Y}\left(t\right)\right)\right] \tag{43}$$

represents the *centralized drift vector*.

Taking into account (40) the centralized drift vector (43) can be expressed as

$$c_i^0\left(\mathbf{Y},t\right)=A_i+B_{im}Y_m^0+C_{imn}Y_m^0Y_n^0+D_{imnp}Y_m^0Y_n^0Y_p^0 \tag{44}$$

where

$$\begin{aligned} A_i &= -f_{imn}\kappa_{mn}-g_{imnp}\kappa_{mnp}-3g_{imnp}\kappa_{mn}\mu_p \\ B_{im} &= e_{im}+2f_{imn}\mu_m+3g_{imnp}\mu_n\mu_p \\ C_{imn} &= f_{imn}+3g_{imnp}\mu_p \\ D_{imnp} &= g_{imnp} \end{aligned} \tag{45}$$

Due to the presence of cubic non-linear terms the set of equations (41) for mean values is not closed. It involves unknown second- and third-order joint centralized statistical moments of the state variables and are given as

$$\frac{d}{dt}\mu_i\left(t\right)=e_{im}\mu_m+f_{imn}\mu_m\mu_n+g_{imnp}\mu_m\mu_n\mu_p-A_i \tag{46}$$

Equations for the second- and higher-order joint statistical moments are obtained with the aid of Itô differential rule. Due to the polynomial non-linearities the moment equations form an infinite hierarchy [5]. In general, if the non-linear terms involve r$^{\text{th}}$ order polynomials the equations for moments up to n$^{\text{th}}$ order involve the moments of orders up to *n-1+r*. In particular, in the case under consideration the equations for second- and third-order moments involve fourth- and fifth-order moments and equation for fourth-order moments include sixth-order moments as follows

$$\frac{d}{dt}\kappa_{ij}\left(t\right)=2\left\{B_{im}\kappa_{mj}\right\}_s+2\left\{C_{imn}\kappa_{mnj}\right\}_s+2\left\{D_{imnp}\kappa_{mnpj}\right\}_s+\left\{b_ib_j\right\}_s \tag{47}$$

$$\frac{d}{dt}\kappa_{ijk}\left(t\right)=3\left\{A_i\kappa_{jk}\right\}_s+3\left\{B_{im}\kappa_{mjk}\right\}_s+3\left\{C_{imn}\kappa_{mnjk}\right\}_s+3\left\{D_{imnp}\kappa_{mnpjk}\right\}_s \tag{48}$$

$$\frac{d}{dt}\kappa_{ijkl}\left(t\right)=4\left\{A_i\kappa_{jkl}\right\}_s+4\left\{B_{im}\kappa_{mjkl}\right\}_s+4\left\{C_{imn}\kappa_{mnjkl}\right\}_s+4\left\{D_{imnp}\kappa_{mnpjkl}\right\}_s+6\left\{b_ib_j\kappa_{kl}\right\} \tag{49}$$

The moment hierarchy needs to be truncated at some level in order to solve equations (46-49). The technique of cumulant - neglect closure (CNC) can be applied in order to implement the truncation procedure [6]. If the equation set is to be truncated at the fourth-order moment level, all fifth- and sixth-order moments must be expressed in terms of lower-order moments. The CNC approximations for the fifth- and sixth-order moments are expressed as

$$\kappa_{ijklm}\left(t\right)=10\left\{\kappa_{ij}\left(t\right)\kappa_{klm}\left(t\right)\right\}_s \tag{50}$$

$$\kappa_{ijklmn}\left(t\right)=15\left\{\kappa_{ij}\left(t\right)\kappa_{klmn}\left(t\right)\right\}_s+10\left\{\kappa_{ijk}\left(t\right)\kappa_{lmn}\left(t\right)\right\}_s-30\left\{\kappa_{ij}\left(t\right)\kappa_{kl}\left(t\right)\kappa_{mn}\left(t\right)\right\}_s \tag{51}$$

The non-linear nonstationary stochastic problem (46-49) has no closed-form exact solution. An approximate solution can be obtained numerically by using the CNC technique. However, one has to keep in mind that it would be difficult to assess the convergence of approximate solutions without conducting direct numerical simulation (such as Monte Carlo simulation). The first approximation to the exact solution could be sought by considering the linear form (38). The mean values of the state variables are then equal to zero and the differential equations governing the second-order statistical moments of the state vector, i.e. the covariance matrix with elements expressed as $\kappa_{ij}=E\left[Y_i\left(t\right)Y_j\left(t\right)\right]$, are given as

$$\frac{d\kappa_{ij}}{dt}=2\left\{A_{im}\kappa_{mj}\right\}_s+b_ib_j;\ i,j=1,2,\ldots,5 \tag{52}$$

where $\left\{\ldots\right\}_s$ denotes the symmetrization operation defined as

$$2\left\{A_{im}\kappa_{mj}\right\}_s=2\frac{\sum_{m=1}^{5}\left(A_{im}\kappa_{mj}+A_{jm}\kappa_{mi}\right)}{2} \tag{53}$$

The lateral displacements are approximated as $\overline{w}\left(x,t\right)=\Phi_r\left(x\right)q_r\left(t\right)$ so that the expectation of the displacements is given as $E\left[\overline{w}\left(x,t\right)\right]=E\left[q_r\left(t\right)\right]\Phi_r\left(x\right)$. The variance of the lateral displacement can then be determined as

$$\sigma_w^2\left(x,t\right)=E\left[\overline{w}^2\left(x,t\right)\right]=\kappa_{rr}\left(t\right)\Phi_r^2\left(x\right) \tag{54}$$

Numerical Results

A linear approximate solution is obtained for a system with *8* ropes of mass per unit length $m = 2.11\ kg/m$ each. In this case the upper support is moving upwards at a speed of *8 m/s*. The numerical tests are conducted for system ascending with the acceleration/ deceleration rate of $a = 1.2\ m/s^2$ (jerk $2.4\ m/s^3$) from the lowest landing upwards to the highest level. The kinematic time profile plots are shown in Figure 3.

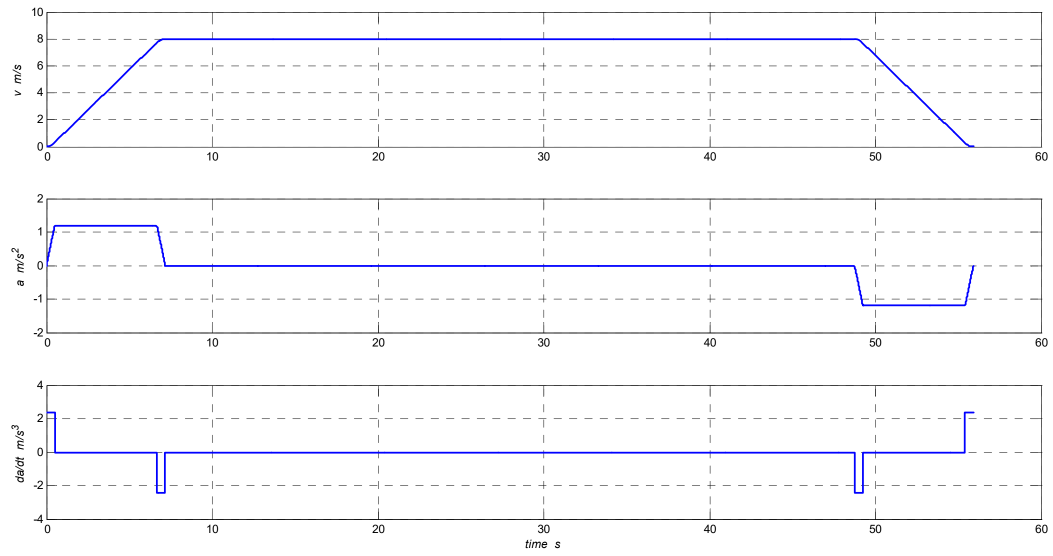

Figure 3. Speed, acceleration/ deceleration and jerk time profiles.

The travel height is $H = 390\ m$ and the host structure is subjected to a sway of frequency $0.1\ Hz$ and amplitude $A_0 = 0.762\ m$ measured at $z_0 = 402.75m$ above the ground floor.

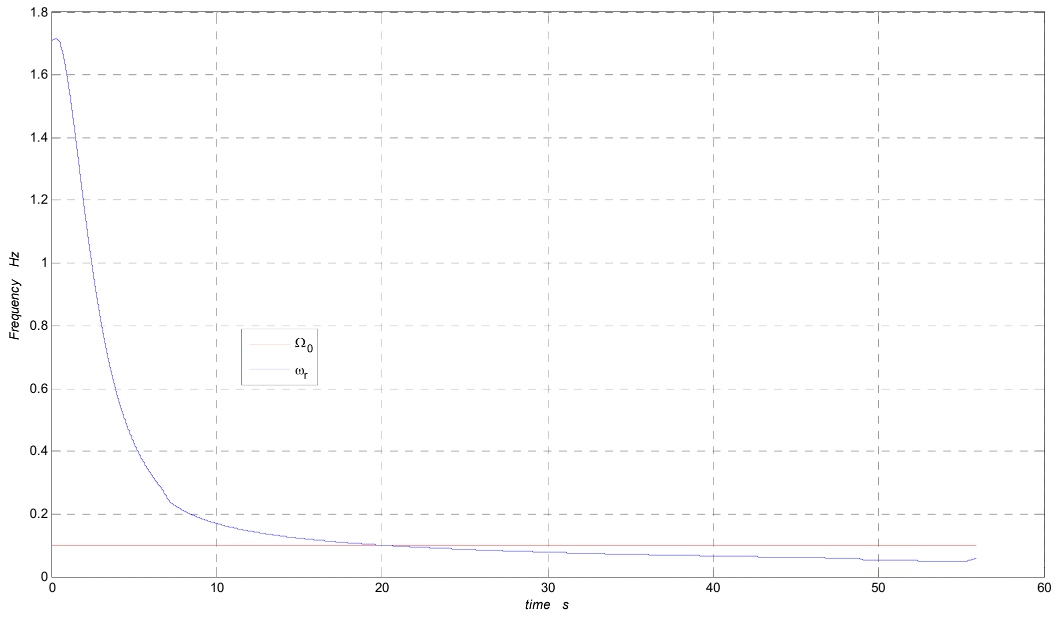

Figure 4. The variation of the fundamental natural frequency (blue curve) together with the frequency of the sway (red line).

The plots shown in Figure 4 demonstrate that the frequency of the structure coincides with the first (fundamental) frequency of the ropes within the time range of about *20 - 21 s*.

An explicit Runge-Kutta (4,5) formula is used to integrate the differential equations (26) and (52) with the damping ratio for the ropes assumed as $\zeta_r = 0.3\%$ and the damping ratio of the filter is taken as $\zeta_f = 0.1\%$. The filter variable α is determined as [4]

$$\alpha = \Omega_0 \left(-\zeta_f + \sqrt{\zeta_f^2 + \frac{\zeta_f \Omega_0^3 A_0^2}{\pi S_0 - \zeta_f \Omega_0^3 A_0^2}} \right) \tag{55}$$

where the constant level of the power spectrum of white noise is assumed as one.

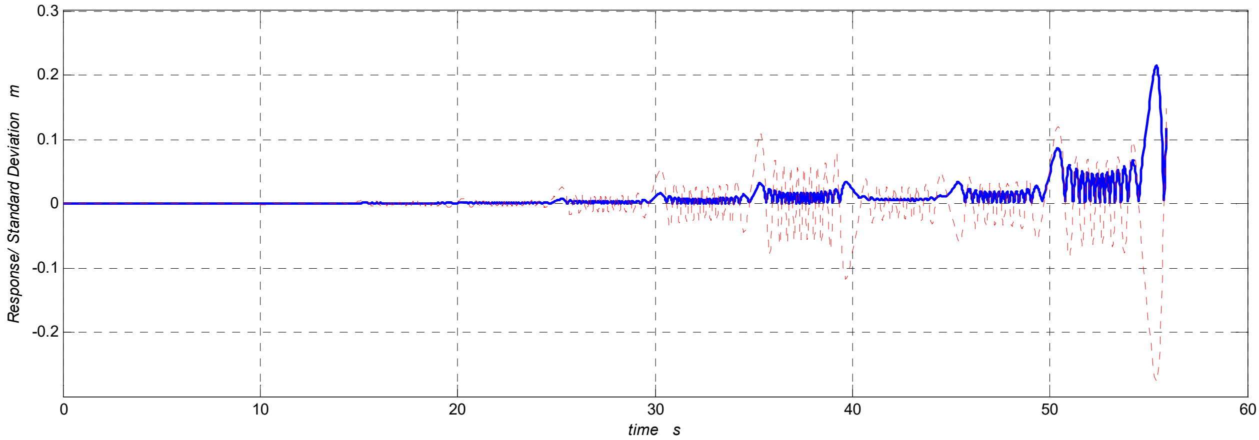

Figure 5. The deterministic dynamic displacements $\overline{w}\left(\frac{L}{2},t\right)$ (red dotted line) and standard deviation (blue solid line) of the displacements $\sigma_{\overline{w}}\left(\frac{L}{2},t\right)$ vs. time.

The plots shown in Figure 5 represent the deterministic dynamic displacements $\overline{w}\left(\frac{L}{2},t\right)$ (red dotted curve) corresponding to motion $w_0(t)$ treated as harmonic process of frequency Ω_0 and the standard deviation of the displacements $\sigma_{\overline{w}}\left(\frac{L}{2},t\right)$ (blue line) determined using the scenario that $w_0\left(t\right)$ has a narrow banded time variation with the centre frequency Ω_0. The deterministic curve demonstrates that a passage through the fundamental resonance takes place and is sustained during the travel. The stochastic solution represented by the standard deviation shows the statistical scatter of the response.

Conclusions

The equations of motion of a vertical rope moving at speed within a tall host structure derived in this paper accommodate the nonlinear effects of rope stretching. This model is used to determine the response of the system under the load due to the sway of the host structure. The loading conditions are represented by a narrow-banded stochastic process. The demand is that the function representing the motion of the structure is twice differentiable and the time derivatives are continuous. Thus, its stochastic variation is represented by filtering the Gaussian white noise through a first-order filter followed by filtering through a second-order filter. Due to the nonlinear effects equations for the second- and higher-order joint statistical moments, obtained with the aid of Itô differential rule, form an infinite hierarchy. In order to truncate the hierarchy of moment equations various closure techniques can be applied. However, the convergence of closure approximations is difficult to assess and the first approximation to the solution is determined by considering a linearized stochastic model. The linear approximation then leads the determination of covariance matrix with its elements showing the statistical scatter of the response of the system.

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